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Interaction of nuclear radiation and particles with matter

Pavia, march 2005

Contents

0.1 Some units

$$e^- = 1.6\ 10^{-19}\ \ \mathrm{C}$$

$$m_e = 9.11\ 10^{-28}\ \ \mathrm{g} = 9.11\ 10^{-31}\ \ \mathrm{Kg}$$

$$1\ \mathrm{eV} = 1.6\ 10^{-19}\ \ \mathrm{Joule}$$

$$c = 2.997\ 10^{8}\ \ \mathrm{m/s}$$

$$m_e \rightarrow m_e c^2 = \frac{9.1\ 10^{-31}\mathrm{Kg}\,(2.997)^2\,10^{16}\ (\mathrm{m/s})^2}{1.6\ 10^{-19}\ \mathrm{J}} = 51\frac{10^{-31}10^{16}}{10^{-19}}$$

$$= 51\ 10^4\mathrm{eV} = 0.511\ \mathrm{MeV}$$

Often the masses are measured in energy

electron m_e = 0.511 MeV
proton m_p = 938.28 MeV
neutron m_n = 939.55 MeV
1 AMU = 931.48 MeV

0.2 Kinematics

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \frac{mc^2}{\sqrt{1 - \beta^2}}$$

$$p = \frac{mv}{\sqrt{1 - \beta^2}} = \frac{mc\frac{v}{c}}{\sqrt{1 - \beta^2}} = \frac{mc\beta}{\sqrt{1 - \beta^2}}$$

$$E^2 = m^2c^4 + p^2c^2$$

In nuclear and radiation physics often one uses the "natural" units

$$c = 1 \ , \quad \text{masses and energies in MeV}$$

$$mc^2 \longrightarrow m \ , \quad p/c \longrightarrow p$$

$$E = \frac{m}{\sqrt{1 - \beta^2}} \quad \text{MeV}$$

$$p = \frac{mc^2\beta/c}{\sqrt{1 - \beta^2}} = \frac{m\beta}{\sqrt{1 - \beta^2}} \quad \text{MeV/c}$$

$$E^2 = m^2 + p^2$$

The gamma factor:

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = \frac{E}{mc^2}$$

$(\gamma - 1)$ **is a measure of the kinetic energy of the particle in units of its rest mass.**

Kinematics

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \frac{mc^2}{\sqrt{1-\beta^2}} = \frac{m}{\sqrt{1-\beta^2}}$$

$$p = \frac{mv}{\sqrt{1-\beta^2}} = \frac{mc\frac{v}{c}}{\sqrt{1-\beta^2}} = \frac{mc^2\frac{\beta}{c}}{\sqrt{1-\beta^2}}$$

$$p = \frac{\frac{m\beta}{c}}{\sqrt{1-\beta^2}} \quad \mathrm{MeV/c} \to pc = \frac{m\beta}{\sqrt{1-\beta^2}}$$

Energies and masses are in MeV, momenta in MeV/c. $E_{\rm tot} \equiv E$ **the energy is the total one!**

$$E_{\rm tot} = E_{\rm kin} + m = \frac{m}{\sqrt{1-\beta^2}} \to \beta = \sqrt{1 - \frac{m^2}{E_{\rm tot}^2}} = \frac{pc}{E_{\rm tot}}$$

Example: the velocity of 1 MeV electron:

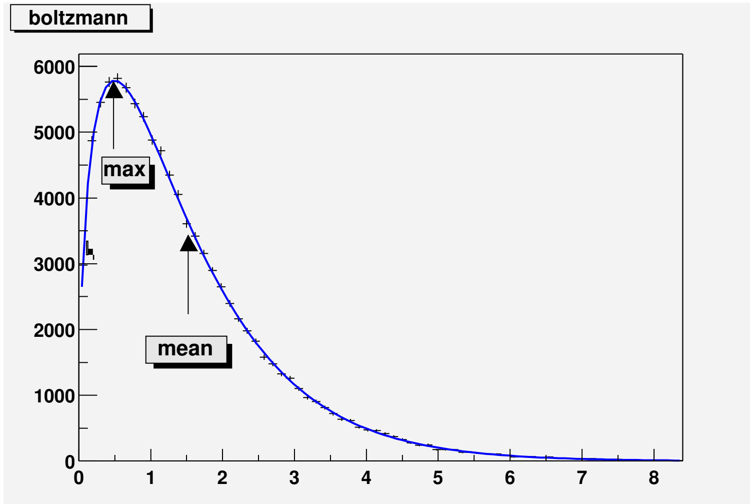
$$\beta = \sqrt{1 - \frac{0.511^2}{(1+0.511)^2}} = 0.94$$

.. is 0.94 times the light velocity (rel. part.)

Example: the velocity of 1 MeV proton:

$$\beta = \sqrt{1 - \frac{938.28^2}{(1+938.28)^2}} = 0.046\ , \quad p \simeq m\beta = 43.2\ \ MeV/c$$

.. is $\simeq 4\%$ **of the light velocity**
(non relativistic particle)



0.3 The Boltzmann distribution

$$f(v)\,\mathrm{d}^3v = \frac{n}{(2\pi v_t^2)^{3/2}}\exp(-v^2/2v_t^2)\ \mathrm{d}^3v$$

$$v_t = \sqrt{\frac{kT}{m}}$$

Energy distribution of a particle when v is a vector of gaussian components with mean velocity $\sqrt{kT/m}$
Most probable energy: $\frac{1}{2}kT$
Mean energy: $\frac{3}{2}kT$
Velocity variance (diffusion): $v_t^2 = \frac{kT}{m}$
When the variable is the kinetic energy

$$\frac{1}{2}mv^2 = E\ , \qquad \mathrm{d}^3v = 4\pi v^2\,\mathrm{d}v\ , \qquad mv\,\mathrm{d}v = \mathrm{d}E$$

$$n(E)\,\mathrm{d}E = \frac{2\pi n}{(\pi kT)^{3/2}}\,E^{1/2}\exp(-E/kT)\,\mathrm{d}E\ . \qquad (1)$$

0.4 The low energy limit

In nuclear physics often one adopts the "natural" units: $c = 1$ and the relativistic formulas. They are very useful even in the low energy limit (contrarily to a widespread belief)

Boltzmann constant: $k = 1.380662\,10^{-16}$ **erg** $^0K^{-1}$ **is often used in energy units:**

$$k = 8.617\ 10^{-11}\ \text{MeV}\ ^0K^{-1} = \frac{1\ \text{eV}}{11\,604\ ^0K} \simeq \frac{1\ \text{meV}}{11.6\ ^0K}$$

that is $1/k \simeq 11.6$ **Kelvin per meV (millielectronVolt).**

Room temperature: $290/11.6 \simeq 25$ **meV**

Low energy limit

$$E_{kin} \equiv E_k = \frac{1}{2}mv^2 = \frac{1}{2}mc^2\frac{v^2}{c^2} = \frac{1}{2}m\beta^2$$

$$\beta = \sqrt{\frac{2E_k}{m}}$$

Example: find the proton velocity at 38^0 **C.**

$$E_k = \frac{1}{2}kT = (273 + 38)/(2 \cdot 11.6) = 13.4\ \text{meV}$$

$$\beta = \sqrt{\frac{2 \cdot 0.0134}{938\ 10^6}} = 5.4\ 10^{-6}\ \text{in units c (1618 m/s)}$$

0.5 Quantum wavelength

$$\lambda = \frac{h}{p} = \frac{hc}{pc} \qquad (1)$$

$$\lambda\!\!\!/ = \frac{\hbar\ c}{pc}$$

$$\hbar\ c = 197.3 \text{ MeV fm} \quad (1 \text{ fm} = 10^{-13} \text{ cm})$$

Example: the 1 MeV neutron wavelength

$$\beta = \sqrt{\frac{2E_k}{m}} = 0.046$$

$$pc = \frac{m\beta}{\sqrt{1-\beta^2}} = 43.35 \text{ MeV} \ , \quad p = 43.35 \text{ MeV/c}$$

$$\lambda = 2\pi\frac{197.3}{43.35} \text{ fm} \ = 28.5\ 10^{-13} \text{ cm}$$

We obtain the dimensions of the nucleus. Conclusion: MeV is the order of magnitude of the nuclear binding energies.

Golden Rule: wavelengths (dimensions of the physical objects) and energies are related by (1)

Quantum wavelength

Example: the 0.025 neutron wavelength (room temperature)

$$\beta = \sqrt{\frac{2 \times 0.025}{939.55 \cdot 10^6}} = 7.29 \cdot 10^{-6} \simeq 2\,200 \mathrm{m/s}$$

$$pc = m\beta = 939.55 \cdot 7.29 \cdot 10^{-6} = 6.85 \cdot 10^{-3}$$

$$\lambda = 2\pi \frac{197.3}{6.85\,10^{-3}} \ \mathrm{fm} \ \ = 1.81 \ 10^{-8} \ \mathrm{cm}$$

We obtain the dimensions of the atom.

0.6 Massless particles

$$E^2 = p^2c^2 + m^2c^4 \xrightarrow{m=0} E = pc$$

$$p = E$$

$$\lambda = 2\pi \frac{\hbar\ c}{pc} = 2\pi \frac{\hbar\ c}{E} \frac{\text{MeV fm}}{\text{MeV}}$$

Example: wavelength of 88 keV photons

$$\lambda = 2\pi \frac{197.3}{0.088} = 1.41 \cdot 10^{-9} \text{ cm}$$

This is the wavelength of the *k*-electrons, coming from the inner atomic shells

$$\lambda = \begin{cases} 2\pi \frac{\hbar\ c}{pc} & \text{heavy particles} \\ 2\pi \frac{\hbar\ c}{E} & \text{massless particles} \end{cases}$$

0.7 Atomic density

If A is the mole and N_A the Avogadro's number, the number of atoms N/cm^3 for a substance of density ρ is given by:

$$N = \frac{\rho\, N_A}{A} \qquad \left[\frac{\text{atoms}}{\text{cm}^3}\right]$$

1 amu $= 1.66053\ 10^{-24}\text{g} = 931.481\ \ \text{MeV}$

The density ρ for gases:

$$pV = \frac{M}{A} RT \ , \qquad R = 0.0821 \frac{\text{atm}}{\text{mole}\ ^0\text{K}}$$

$$\rho(\text{kg/m}^3) = \rho(\text{g/l}) = 1000\,\rho(\text{g/cm}^3) = \frac{M}{V} = 12.18\,\frac{A}{T}\,p(\text{atm})$$

The Avogadro number:

$$\frac{1}{1\ \text{amu}} = 6.022 \cdot 10^{23} \to N_A$$

Example: Sodium

$$\frac{0.97\ 6.022\ 10^{23}}{22.99} = 2.54\ 10^{22}\ \text{atoms/cm}^3$$

Example: Na Cl

$$\frac{2.17\ 6.022\ 10^{23}}{58.44} = 2.24\ 10^{22}\text{atoms/cm}^3$$

0.8 Nuclear Reactions

$$a + b \to c + d$$

$$E_k(a) + E_k(b) + m_a + m_b = E_k(c) + E_k(d) + m_c + m_d$$

Conservation laws:

- **nucleon number conservation**
- **charge conservation**
- **momentum conservation**
- **energy conservation**

Q-value

$$\begin{aligned} Q &= (m_a + m_b) - (m_c + m_d) \\ &= [E_k(c) + E_k(d)] - [E_k(a) + E_k(b)] \end{aligned}$$

$Q > 0$ exothermic reaction, lighter final masses

$Q < 0$ endothermic reaction, heavier final masses

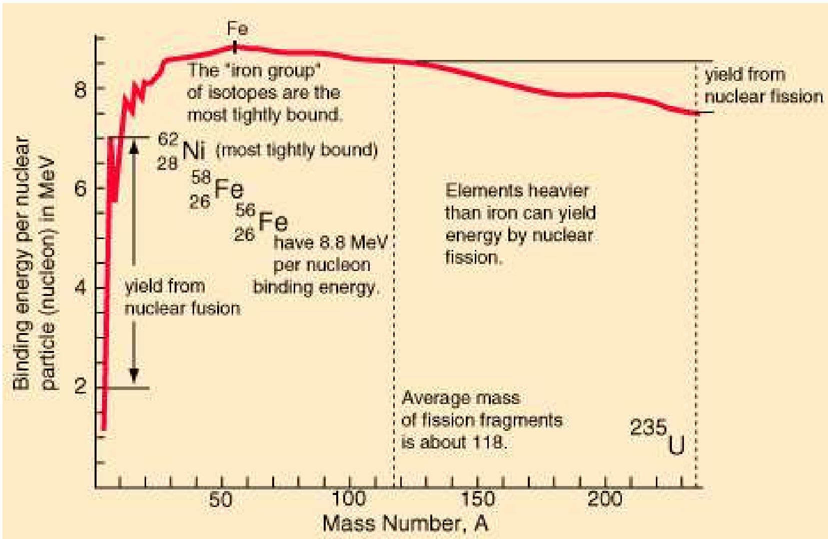
The relativistic energy conservation applied to the decay of a particle M (for example into 2 particles) defines the binding energy ΔM:

$$M = m_1 + m_2 + E_k \to M > m_1 + m_2$$

$$\Delta M = M - (m_1 + m_2) = \text{Binding Energy}$$

Binding energy for a nucleus of mass M_A:

$$\Delta = Zm_p + Nm_n - M_A$$



0.9 Binding Energy and Mass excess

Binding Energy or Mass Defect: the energy spent to create the bound system
Mass excess: binding energy on the ^{12}C scale
Example: Calculate the binding energy of the external neutron of the ^{13}C nucleus.

$$m_n = \frac{939.55}{931.48} = 1.008664 \ , \qquad {}^{12}C + m_n = 13.008664$$

$$^{13}C = 13.00335 \quad \text{experimental value}$$

$$\Delta_n = 13.008664 - 13.00335 = 5.31 \cdot 10^{-3}$$

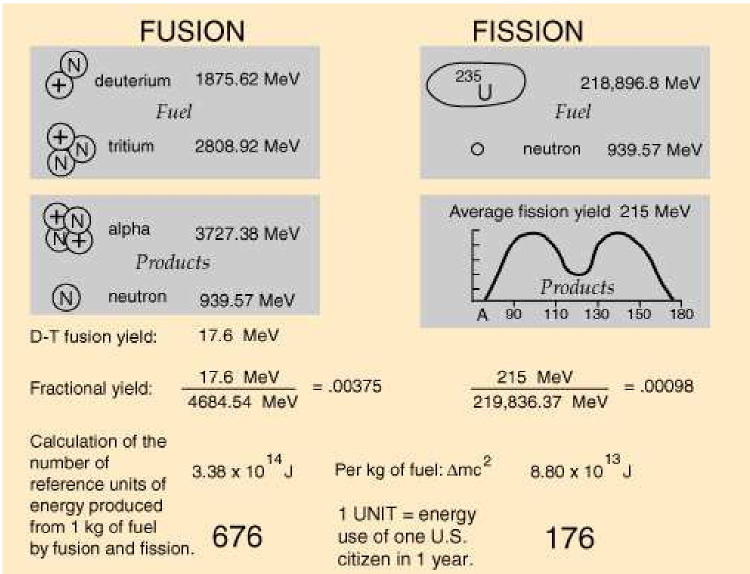
$$\Delta_n(\text{MeV}) = 5.31 \cdot 10^{-3} \cdot 931.48 \quad \text{MeV} = 4.95 \quad \text{MeV}$$

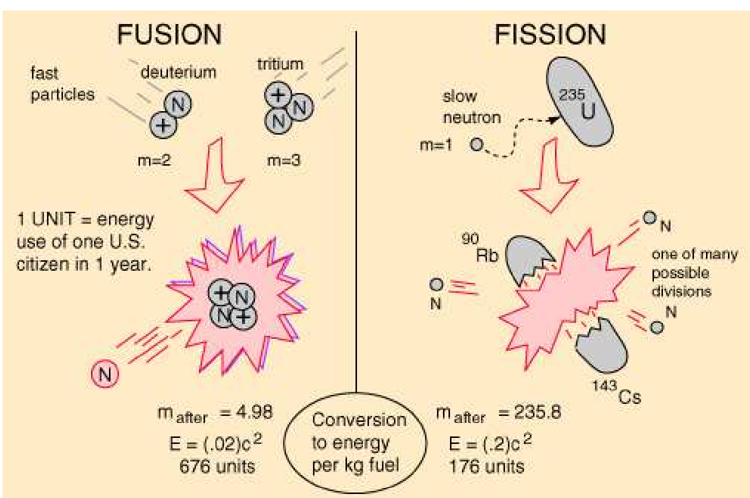
From the mass excess tables: $\Delta m(^{13}C) = 3.125$ **MeV**

$$\Delta m = (M - A)\, 931.48 \quad \rightarrow \quad M = \frac{\Delta m}{931.48} + A$$

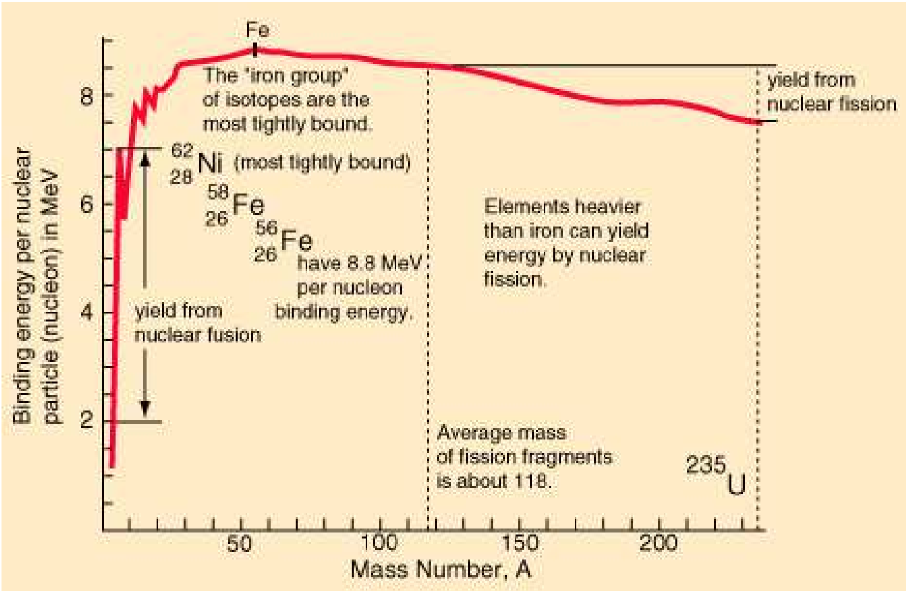
Hence: $M(^{13}C) = 3.125/931.48 + 13 = 13.00335$

0.10 Nuclear Fusion and Fission





Nuclear Fusion



Two light nuclei give a heavier and more stable nucleus

$$d + d \rightarrow t + p$$

$$^2H +^2 H \rightarrow^3 H + p$$

Deuteron binding energy:

$$938.28 + 939.55 - 2.0136 \times 931.5 \simeq 2.23 \text{ MeV}$$

Tritium binding energy = 8.48 MeV

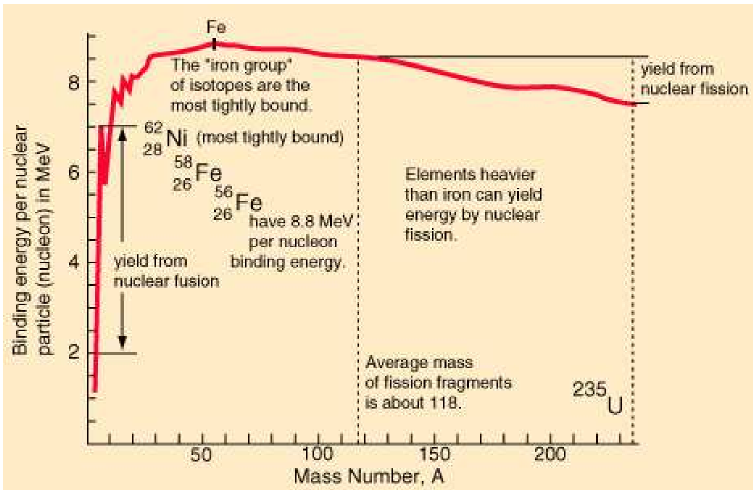
Reaction Q-value:

$$2m_p+2m_n-2E_b(^2H)-m_p-2m_n-m_p+E_b(^3He) = E_b(^3He)-2E_b(^2H)$$

$$8.48 - 2 \times 2.23 = 4.42\text{MeV}$$

This energy excess transforms in the kinetic energy of tritium and proton

Nuclear Fission



A heavy nucleus breaks-up into two (or more) lighter nuclei

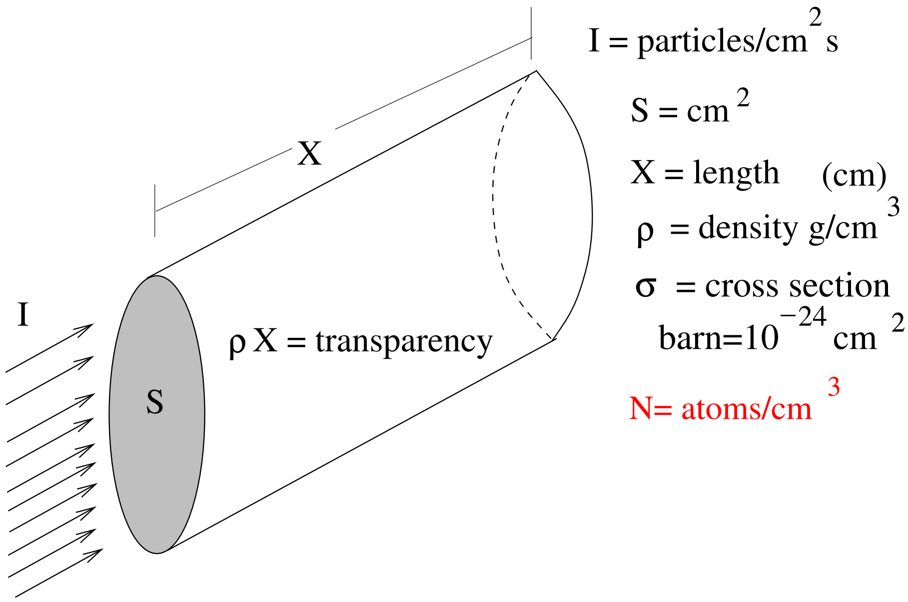
$$^{235}U \rightarrow ^{135}A + ^{100}A \quad \text{average values}$$

Binding energies:

$$\Delta(^{235}U) = 235 \times 7.5 = 1762 \text{ MeV}$$
$$\Delta(^{135}A + ^{100}A) = 235 \times 8.4 = 1974 \text{ MeV}$$

Q-value: 212 MeV

0.11 Cross Section



$$\frac{\#\ \text{collisions}}{\text{s}} = \sigma I S X N = \sigma I S X \frac{\rho N_A}{A} \qquad (2)$$

collisions/cm^{2}s $= \sigma I \rho X \frac{N_A}{A} = \sigma I N X \equiv \Sigma I X$

$$\rho X \ \ (\text{g/cm}^2) = \text{transparency}$$

$\Sigma = \sigma N = \sigma \rho N_A / A \ (\text{cm}^{-1}) = \text{macroscopic cross section}$

Example: ^{12}C, $\sigma = 2.6$ **barn**, $I = 5 \cdot 10^8$ **neutrons/s cm^2**, $X = 0.05$ **cm**

$$\sigma I \rho X N_A / A = 2.6 \cdot 10^{-24}\ 5 \cdot 10^8\ 1.60 \cdot 0.05 \cdot 0.602 \cdot 10^{24}/12$$
$$= 5.2 \cdot 10^6\ \text{int/cm}^2\ \text{s}$$

Interaction probability: $\sigma \rho X \frac{N_A}{A} = \frac{5.2\ 10^6}{5 \cdot 10^8} = 1\ 10^{-2}$

0.12 Mean Free Path

$$\frac{\text{collisions}}{\text{cm}^2\ \text{s}} = [I(x) - I(x + \mathrm{d}x)] = -\,\mathrm{d}I = \sigma I N\ \mathrm{d}x = \Sigma I\,\mathrm{d}x$$

We obtain the equation

$$\frac{\mathrm{d}I}{\mathrm{d}x} = -I\,\Sigma$$

which has as a solution:

$$I(x) = I_0\ \mathrm{e}^{-\Sigma\,x} \tag{3}$$

the exponential attenuation of the beam. The important quantities related to this solution are:

$$\text{surviving probability}: \ I(x)/I_0 = \mathrm{e}^{-\Sigma\,x}$$

$$\text{“death}'' \text{ probability}: \ [I_0 - I(x)]/I_0 = 1 - \mathrm{e}^{-\Sigma\,x}$$

probability density for a path x: $p(x) = \Sigma\,\mathrm{e}^{-\Sigma\,x}$

mean free path (cm):

$$\lambda = \int x p(x)\ \mathrm{d}x = \int_0^\infty x\,\Sigma\,\mathrm{e}^{-\Sigma\,x}\,\mathrm{d}x = \frac{1}{\Sigma}$$

Monte Carlo mean free path simulation ($0 \leq RANDOM \leq 1$):

$$1 - \mathrm{e}^{-\Sigma\,x} = RANDOM \ \ \rightarrow \ \ x = -\frac{1}{\Sigma}\ln(1 - RANDOM)$$

0.13 Molecules

$$R = \frac{\text{events}}{s} = \sigma N\, ISX = \sigma\,\frac{\rho N_A}{A}\, ISX = \Sigma\, ISX$$

How to calculate cross sections σ_T or the interaction rate R for molecules and compounds, starting from those of the elements?

Molecule M= $X_m Y_n \qquad A = mA_x + nA_y$

Atoms simply sum-up (cm^2)

$$\sigma_T = m\sigma_x + n\sigma_y$$

Macroscopic cross sections sum-up (cm^{-1})

$$\Sigma_T = \frac{\rho N_A}{A}\, m\sigma_x + \frac{\rho N_A}{A}\, n\sigma_y = \frac{N_x}{N} N\,\sigma_x + \frac{N_y}{N} N\,\sigma_y = m\Sigma_x + n\Sigma_y$$

The event rate can be written
independently of the density!

$$\mu = \frac{\sigma\, N}{\rho} = \frac{\Sigma}{\rho} \qquad \text{dimensions} \left[\frac{\text{cm}^2}{\text{g}}\right] \tag{4}$$

$$\begin{aligned} \frac{\Sigma}{\rho} &= \frac{N_A}{mA_x + nA_y} m\,\sigma_x + \frac{N_A}{mA_x + nA_y} n\,\sigma_y \\ &= \frac{mA_x}{mA_x + nA_y}\,\frac{N_A}{A_x}\sigma_x + \frac{nA_y}{mA_x + nA_y}\,\frac{N_A}{A_y}\sigma_y \end{aligned} \tag{5}$$

$$\frac{\Sigma}{\rho} = w_x \left(\frac{\Sigma}{\rho}\right)_x + w_y \left(\frac{\Sigma}{\rho}\right)_y$$

where w_x and w_y are the molecular (weight) fractions $w_x = mA_x/(mA_x + nA_y)$ (H_2O, $w_H = 2/18, w_O = 16/18$)
If one uses Σ/ρ instead of Σ the thickness X must be expressed as the transparency ρX.

0.14 Mixtures

In a mixture the number of atoms of each species (x, y, ...) is related to the weight fractions (w_x, w_y, ...):

$$R = \frac{\text{events}}{s} = \left[\sigma_x \, w_x \rho \, \frac{N_A}{A_x} + \sigma_y \, w_y \rho \, \frac{N_A}{A_y} \right] \, ISX$$

This formula defines the quantity Σ/ρ:

$$\frac{R}{\rho} = \frac{\text{events cm}^3}{\text{g s}} = \left[w_x \, \frac{\Sigma_x}{\rho_x} + w_y \, \frac{\Sigma_y}{\rho_y} \right] \, ISX$$

formally identical to the formula for molecules:

$$\frac{\Sigma}{\rho} = w_x \left(\frac{\Sigma}{\rho} \right)_x + w_y \left(\frac{\Sigma}{\rho} \right)_y \qquad \left[\frac{\text{cm}^2}{\text{g}} \right]$$

For gas mixtures at constant p, T (M is the mass):

$$pV = \frac{M}{A} RT \; , \quad V = \sum_i V_i = \sum_i \frac{w_i M}{A_i} \frac{RT}{p} \; , \quad pV = \sum_i \frac{w_i M}{A_i} RT$$

$$\frac{1}{A} = \sum_i \frac{w_i}{A_i} \quad \longrightarrow \quad \frac{1}{\rho} = \sum_i \frac{w_i}{\rho_i}$$

where ρ_i is the density of the i-th species at the same p and T.

Volume and weight percentages

When mixing gases, often one knows the volume percentages

If one mixes two gases 1 and 2 with volume % α **and** β **and atomic numbers** A_1 **and** A_2**:**

$$V_\alpha = \alpha V = \frac{w_1 M}{A_1}\frac{RT}{p} \; , \quad V_\beta = \beta V = \frac{w_2 M}{A_2}\frac{RT}{p}$$

From the gas law $pV = MRT/A$**:**

$$\frac{\alpha}{A} = \frac{w_1}{A_1} \; , \quad \frac{\beta}{A} = \frac{w_2}{A_2} \quad \rightarrow \quad \frac{\alpha A_1}{A} = w_1 \; , \quad \frac{\beta A_2}{A} = w_2$$

$$\alpha A_1 + \beta A_2 = (w_1 + w_2)A = A$$

Therefore the link between weight w_i **and volume percentages** $\alpha, \; \beta$ **is:**

$$w_1 = \frac{\alpha A_1}{\alpha A_1 + \beta A_2} \; , \quad w_2 = \frac{\beta A_2}{\alpha A_1 + \beta A_2}$$

and these w_i **can be used in the previous formulae as weight percentages.**

Molecules and Mixtures

The density ρ does depend linearly on the atomic weight A.
The cross section of many effects depend linearly on the target atomic number Z

Hence, the average ratio Z/A can be defined as

$$\left\langle \frac{Z}{A} \right\rangle = \sum_i w_i \frac{Z_i}{A_i} = \sum_i \frac{n_i A_i}{\sum_j n_j A_j} \frac{Z_i}{A_i} = \frac{\sum_i n_i Z_i}{\sum_j n_j A_j}$$

However, $\langle I \rangle$ defined in this way is underestimated, because in a compound the electrons are more tightly bound than in free elements.

0.15 Molecules and Mixtures

Apart from the density, that is in terms of number of atoms, a mixture can be thought of as made up of thin layers of pure elements. Hence molecules (compounds) and mixtures can be treated in the same manner (Bragg principle of additivity)

$$\frac{\Sigma}{\rho} = w_x \left(\frac{\Sigma}{\rho}\right)_x + w_y \left(\frac{\Sigma}{\rho}\right)_y \qquad (6)$$

where w_x and w_y are the molecular (weight) fractions for compounds $w_x = mA_x/(mA_x + nA_y)$ $(H_2O,\ w_H = 2/18, w_O = 16/18)$ and fractions by weight for mixtures, where ρ is the density of the mixture.

$\Sigma_x = N\sigma_x$ is the macroscopic cross section calculated using the density of the compound or mixture and the cross section of the species x

$(\Sigma/\rho)_x$ is the macroscopic cross section calculated using both the density of the aggregate where Σ has been measured and the cross section of the species x.
Remember

$$\frac{\text{events}}{\text{s}} = \frac{\Sigma}{\rho}\rho X\, IS$$

if one uses Σ/ρ instead of Σ the thickness X must be expressed as the transparency ρX.

Examples

Absorption cross sections: $H = 3$, $O = 8$ **barn**

1) Calculate the interaction probability per unit time in 1 cm of water.

For the probability: $I = 1/\mathrm{cm}^2\mathrm{s}$, $S = 1$ **cm**

$$\frac{\mathrm{prob}}{s} = \sigma N X = \sigma \rho \frac{N_A}{A} X$$

$$\sigma = 2\,\sigma_H + \sigma_O = 14 \;\mathrm{barn}$$

$$\frac{\mathrm{prob}}{s} = \sigma N X = 14\,10^{-24} \times 1 \times \frac{6.022\;10^{23}}{18} \times 1 = 0.077 \simeq 8\%$$

2) Calculate the interaction probability per unit time in 1m of gas mixture 80% H_2 **and 20%** O_2 **in weight at NTP.**

Densities: $\rho_H = 0.0899$ **mg/cm**^{3}, $\rho_O = 1.428$ **mg/cm**^{3}.

Mixture density:

$$\frac{1}{\rho} = \frac{0.8}{0.0899} + \frac{0.2}{1.428} \quad \rightarrow \quad \rho = 0.1106 \;\mathrm{mg/cm}^3$$

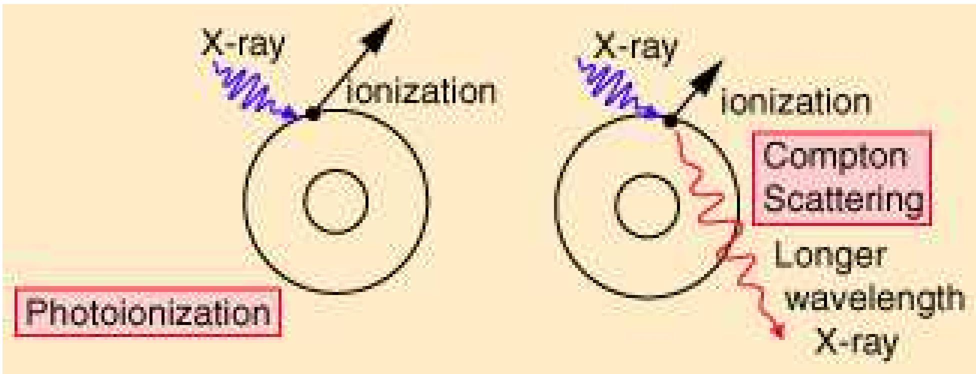
$$P = \frac{\mathrm{prob}}{s} = \left[2\,\sigma_H\, w_H\, \rho \frac{N_A}{A_{H_2}} + 2\,\sigma_O\, w_O\, \rho \frac{N_A}{A_{O_2}} \right] \; I S X$$

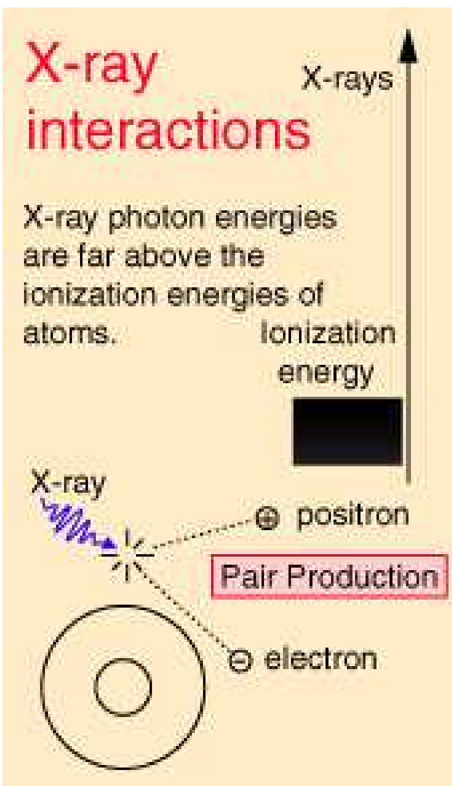
For the probability: $I = 1/\mathrm{cm}^2\mathrm{s}$, $S = 1$ **cm**

$$P \;=\; \left[2 \cdot 3\;10^{-24} \cdot 0.8 \cdot \frac{0.1106\;10^{-3}}{2} + 2 \cdot 8\;10^{-24} \cdot 0.2 \cdot \frac{0.1106\;10^{-3}}{32} \right]$$

$$\times \quad 6.022\;10^{23} \;\times 100 = 0.0166 \simeq 1.7\%$$

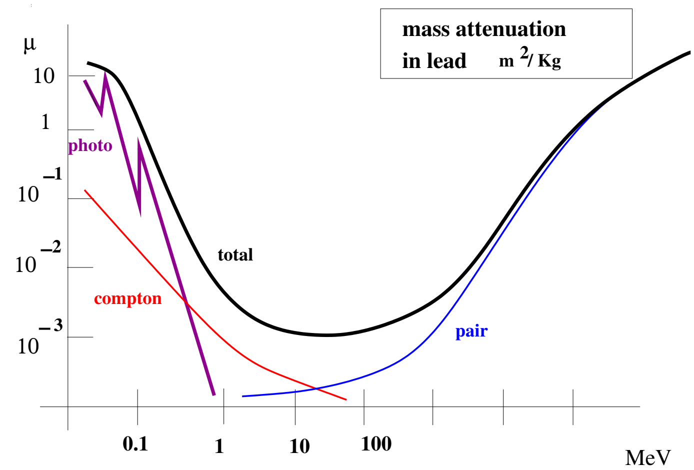
0.16 Gamma Radiation





0.17 Photoelectric effect

Is the dominant process at low energy, in the so.called X-ray domain (X-ray: low gamma with low energy of the order of the atomic transitions)



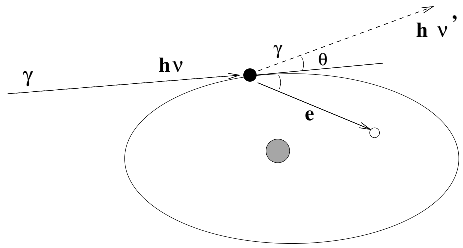
$$h\nu = V_0 + E(e^-)_k$$

V_0 is the extraction potential, E_k is the kinetic energy of the electron.

$$\sigma_{ph} \simeq Z^5 \lambda^{7/2} \propto \frac{Z^5}{E^{7/2}}$$

The photoelectric effect does not happen on the free electron (energy-momentum conservation)
The atom is often deexcites with the emission of a secondary gamma (soft X-ray radiation) or with a low energy electron (Auger electron) when the soft X-ray converts into the atom by the internal photoelectric effect.

0.18 Compton effect



From the energy conservation:

$$h\nu + m_e c^2 = h\nu' + mc^2 \ , \quad \nu = \frac{c}{\lambda} \ , \quad m = \frac{m_e c^2}{\sqrt{1-\beta^2}}$$

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

Output photon energy

$$h\nu' = \frac{h\nu}{1 + \epsilon(1 - \cos\theta)} \ , \qquad \epsilon = \frac{h\nu}{m_e c^2} \tag{7}$$

Recoil electron energy

$$E_e = h\nu - h\nu' = h\nu \, \frac{\epsilon\,(1 - \cos\theta)}{1 + \epsilon\,(1 - \cos\theta)}$$

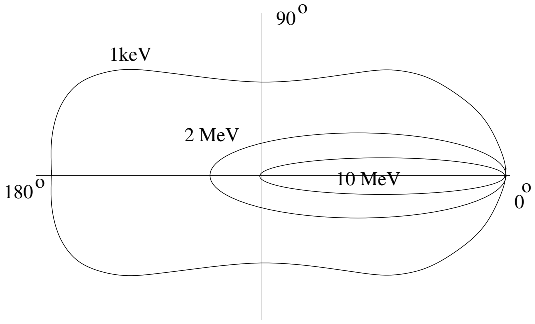
$$E_{\max} = h\nu \frac{2\,\epsilon}{1 + 2\,\epsilon} \ , \quad \theta = 180^0 \quad \text{(Compton edge)}$$

Gamma backscattering energy

$$(h\nu)_{back} = h\nu - E_{\max} = \frac{h\nu}{1 + 2\epsilon}$$

The cross section is given by the Klein-Nishina formula; it decreases by decreasing the energy as $1/(1+\epsilon)$ **and at high energies** ($h\nu \gg m_e c^2$) **the angular distribution is very forward peaked**

0.19 Compton effect. Angular distribution



The angular distribution of the scattered photon becomes strongly forward peaked with increasing the energy.

The angular distribution is given by the famous Klein-Nishina formula:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = Zr_0^2 \left(\frac{1}{1+w(1-\cos\theta)}\right)\left(\frac{1+\cos^2\theta}{2}\right) \times \left(1+\frac{w^2(1-\cos\theta)^2}{(1+\cos^2\theta)[1+w(1-\cos\theta)]}\right)$$

where r_0 is the classical electron radius

$$r_0 = \frac{e^2}{4\pi\epsilon_0 m_e c^2} = 2.817\ 10^{-13}\ \mathrm{cm}$$

$$w = \frac{E_\gamma}{m_e c^2} = \frac{h\nu}{m_e c^2}$$

0.20 Pair production

The reaction has a threshold of $2m_e = 1.022$ MeV:

$$h\nu = \mathrm{e}^+ + \mathrm{e}^- + \text{ recoil}$$

to conserve energy-momentum, the reaction must occur with a third electron or (more often) with a nucleus, which absorb the recoil momentum.
When the recoil is totally absorbed by an electron, one observes two energetic electrons and a a positron.

For relativistic energies the cross section for producing a positron with energy between $(E_+, E_+ + \mathrm{d}E_+)$ is:

$$\sigma_0 = \left[\frac{e^2}{mc^2}\right]^2 \frac{Z^2}{137} \simeq 8 \cdot 10^{-26} \frac{Z^2}{137} \quad [\mathrm{cm}^2]$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E} = \frac{4\sigma_0}{h\nu}\left[\left(w_+^2 + w_-^2 + \frac{2}{3}w_+w_-\right)\ln\left(\frac{183}{Z^{1/3}}\right) - \frac{1}{9}w_+w_-\right]$$

where $w_\pm = E_\pm/(h\nu)$.
At high energies the limit for the total cross section is:

$$\sigma_p \simeq 12\sigma_0$$

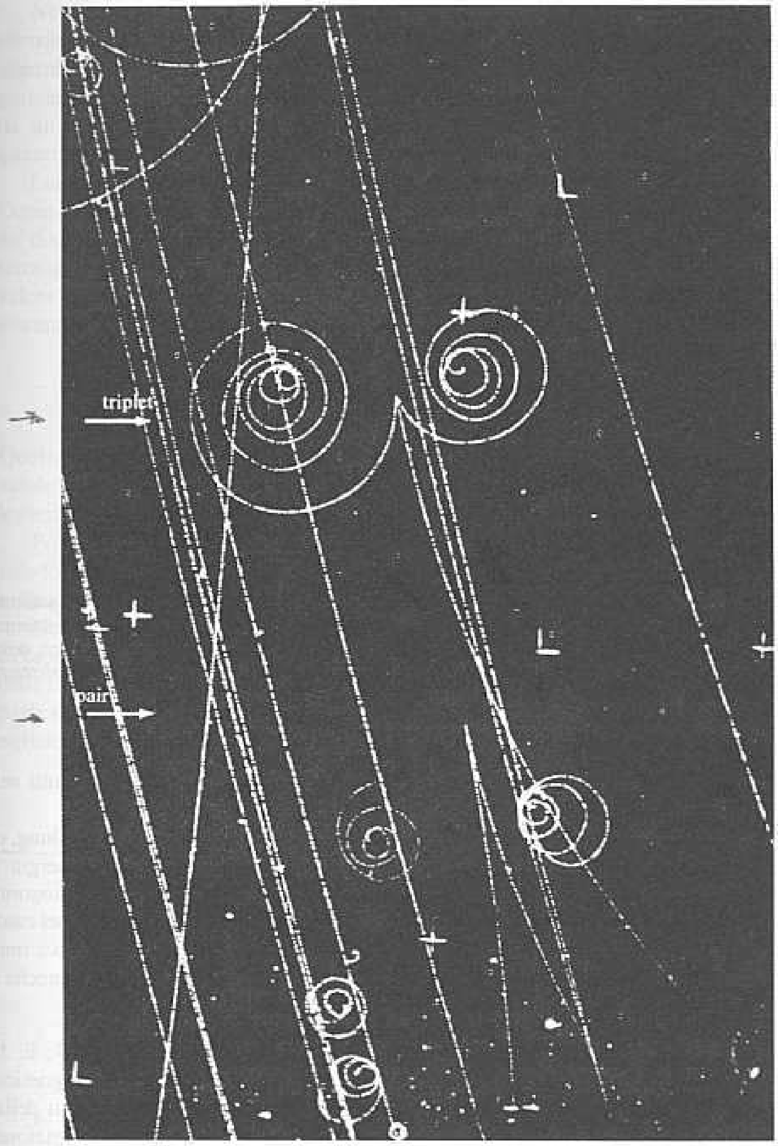


Figura 2.28. Formazione di una coppia elettrone-positrone nel campo di un elettrone (triplettó). Formazione di una coppia nel campo di un protone (coppia). (Camera a bolle a idrogeno). [Foto gentilmente concessa dal Lawrence Radiation Laboratory].

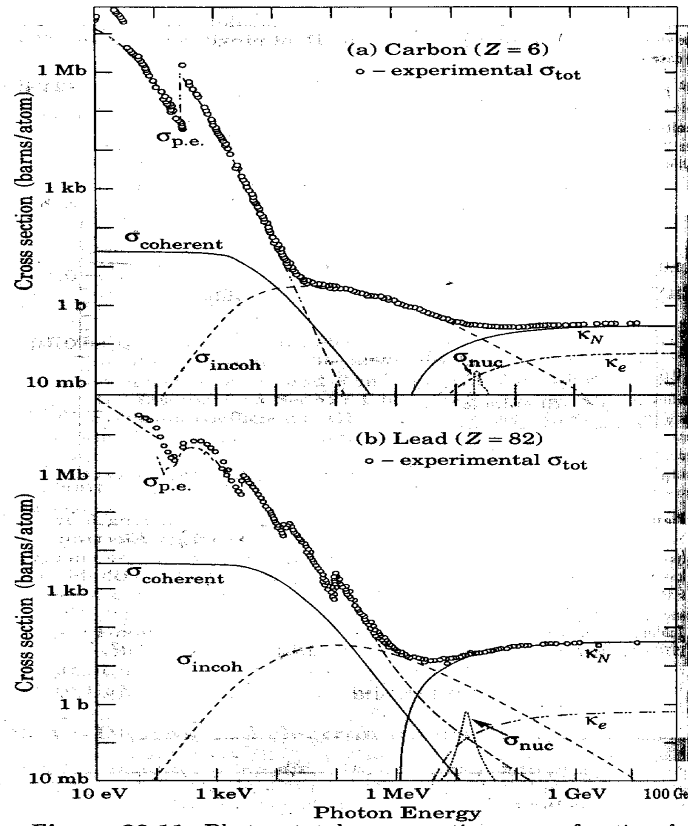


Figure 23.11: Photon total cross sections as a function of energy in carbon and lead, showing the contributions of different processes:

$\sigma_{\text{p.e.}}$ = Atomic photoeffect (electron ejection, photon absorption)

σ_{coherent} = Coherent scattering (Rayleigh scattering—atom neither ionized nor excited)

$\sigma_{\text{incoherent}}$ = Incoherent scattering (Compton scattering off an electron)

κ_n = Pair production, nuclear field

κ_e = Pair production, electron field

σ_{nuc} = Photonuclear absorption (nuclear absorption, usually followed by emission of a neutron or other particle)

From Hubbell, Gimm, and Øverbø, J. Phys. Chem. Ref. Data **9**, 1023 (1980). Data for these and other elements, compounds, and mixtures may be obtained from http://physics.nist.gov/PhysRefData. The photon total cross section is assumed approximately flat for at least two decades beyond the energy range shown. Figures courtesy J.H. Hubbell (NIST).

Carbon and Lead

0.21 Attenuation coefficients

In the case of gamma interaction

$$\Sigma \quad \rightarrow \quad \mu = \mu_{\text{ph}} + \mu_{\text{pp}} + \mu_{\text{c}} = N\sigma \qquad \left[\frac{1}{\text{cm}}\right]$$

Gamma ray intensity:

$$I = I_0\,\mathrm{e}^{-\mu x} = \mathrm{e}^{-\frac{\mu}{\rho}\,\rho x}$$

$$\rho\ X \quad \frac{\text{g}}{\text{cm}^2} \quad \text{transparency}$$

The quantity μ is the attenuation coefficient, so that the intensity $I_0(1-\mathrm{e}^{-\mu\,x})$ is that of the gamma's that made an interaction, not the intensity of the absorbed ones:

- **Photoeffect: γ is absorbed and the photoelectron(s) carry out the energy (total gamma absorption);**
- **Compton effect: γ loses only a part of the primary energy;**
- **Pair Production: the primary γ annihilates into a $e^+\ e^-$ couple, but the subsequent e^+ annihilation produces a $\gamma\ \gamma$ couple, so that part of the primary energy remains in form of electromagnetic radiation**

Sometime the absorption coefficient μ_{ab} is used:

$$W = E\ I\ \mu_{\text{ab}} \qquad \left[\frac{\text{absorbed energy}}{\text{cm}^3\ \text{s}}\right]$$

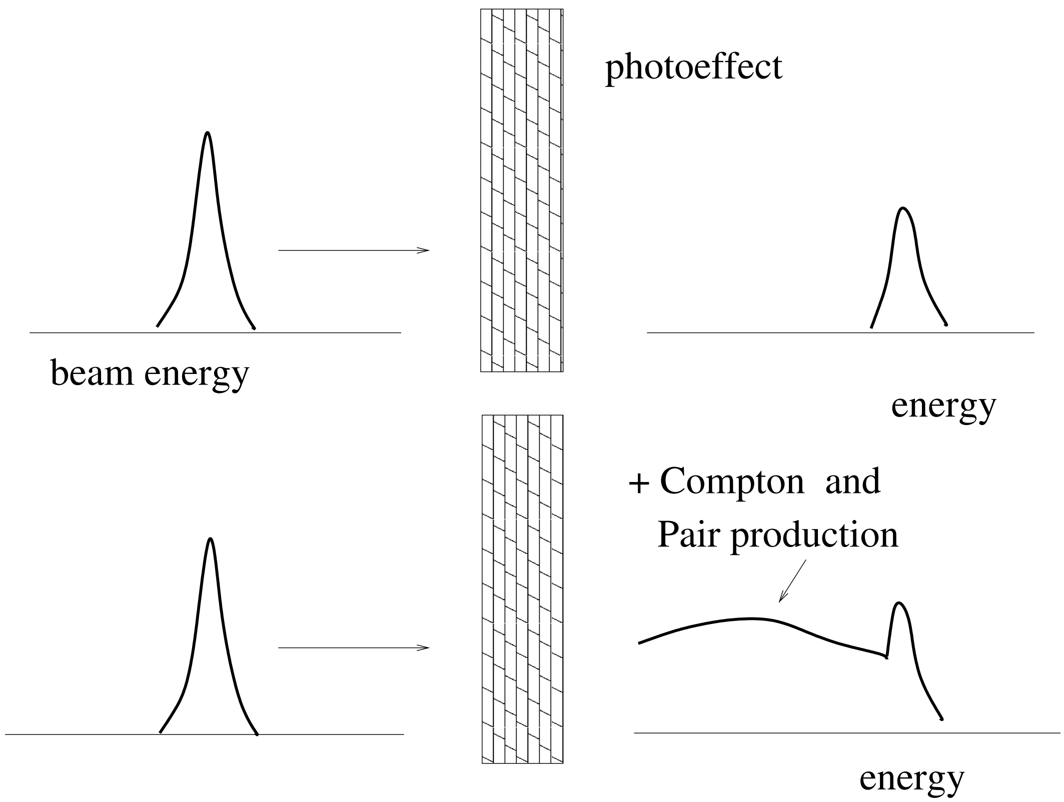
where E is the γ incident energy and I the flux.
It is found experimentally or evaluated by Monte Carlo

0.22 Build-up factor

The uncollided beam

$$I_p = I_0 \ \mathrm{e}^{-\mu x}$$

is the area of the peak at the exit of an absorber. The presence of Compton scattering and pair production fill a tail of lower energy γ to the left of the peak.



The Build-up Factor multiplies the uncollided flux to give the correct total flux (at all the energies) after the absorber:

$$I(x) = I_0 B(\mu x)\, \mathrm{e}^{-\mu x}$$

Build-up factor: example

2 MeV energy γ, $I = 10^6$ γ/cm^2 impinging on a lead screen 10 cm thick. Calculate: a) the uncollided flux b) the outcoming flux

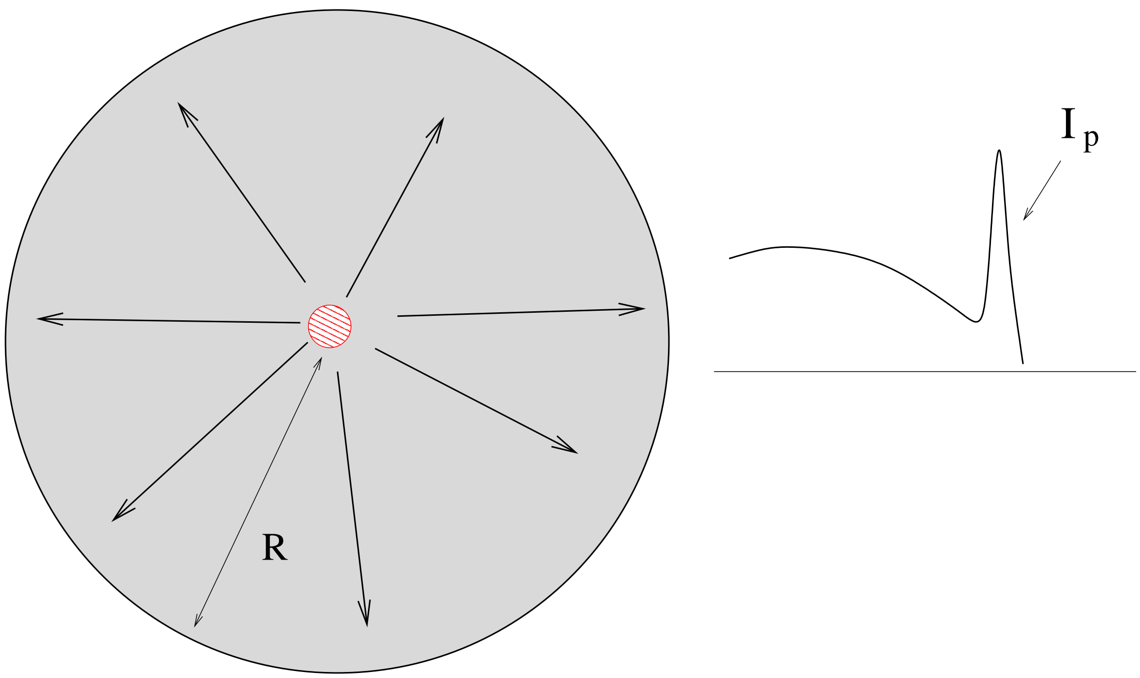
a) From the tables, 2 MeV γ on lead:
$\mu/\rho = 0.0457$ **cm^2/g,** $\rho =$**11.34 g/cm^3**
$\mu = 0.0457 \times 11.34 = 0.518$ **cm^{-1},**
mean free path = $\lambda = 1/\mu = 1.93$ **cm**
$\mu\ X = 0.518 \times 10 = 5.18$ **mean free paths**

$$I_p = 10^6\ \mathrm{e}^{-5.18} \simeq 5.63\ 10^3 \quad \frac{\gamma}{\mathrm{cm^2 s}}$$

b) from the build-up tables: $B(5.18) = 2.78$

$$I(X) = B(\mu X) I_p = 2.78 \times 5.63\ 10^3 = 1.56\ 10^4 \quad \frac{\gamma}{\mathrm{cm^2 s}}$$

Build-up factor: isotropic source



$$I = \frac{S}{4\pi R^2} \qquad \left[\frac{\gamma}{\mathrm{cm}^2\,\mathrm{s}}\right]$$

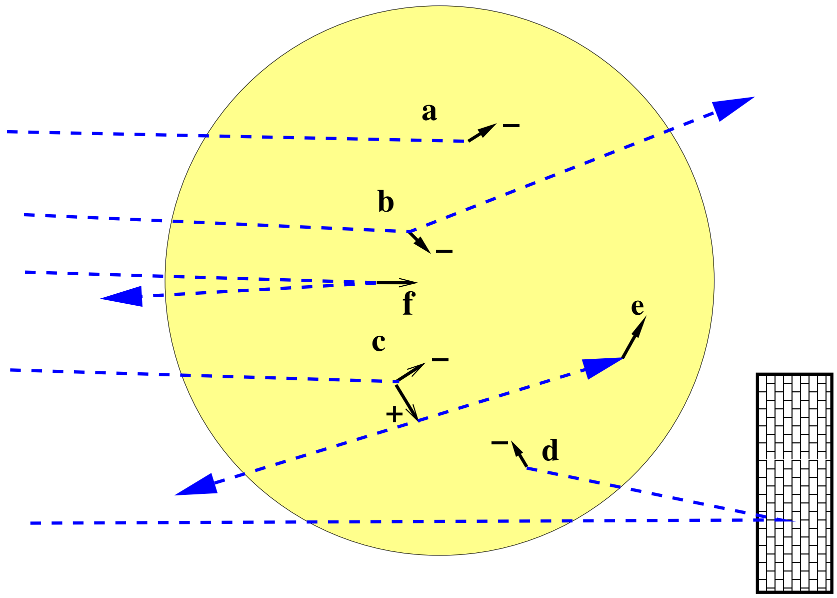
Uncollided flux:

$$I_p = \frac{S}{4\pi R^2}\ \mathrm{e}^{-\mu R}$$

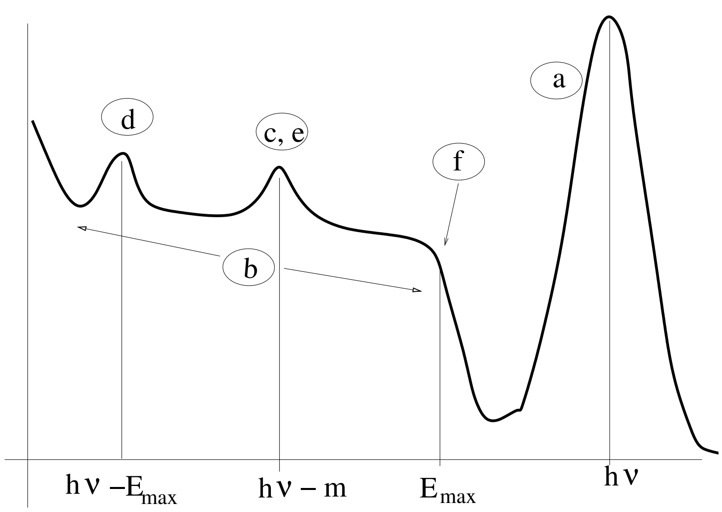
Outcoming flux:

$$I_p = \frac{S}{4\pi R^2} B_R(\mu R)\ \mathrm{e}^{-\mu R}$$

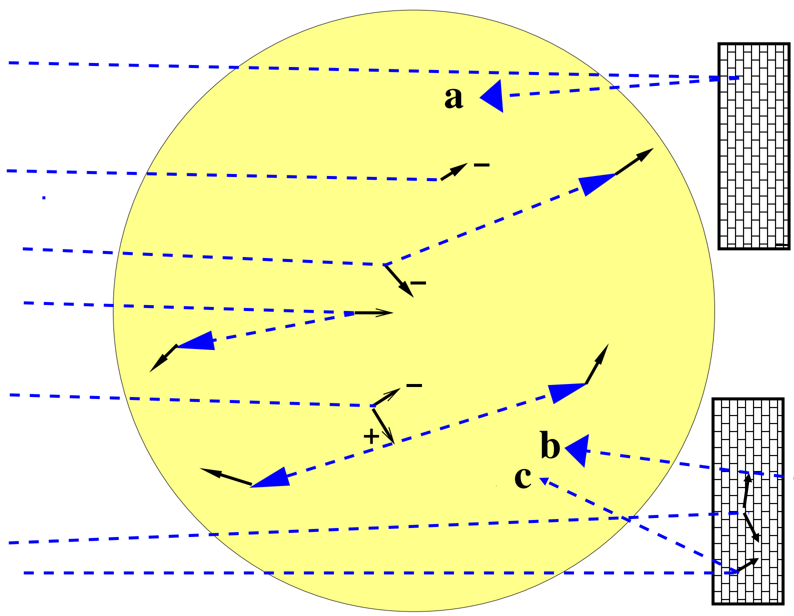
0.23 Gamma spectrum in a small detector



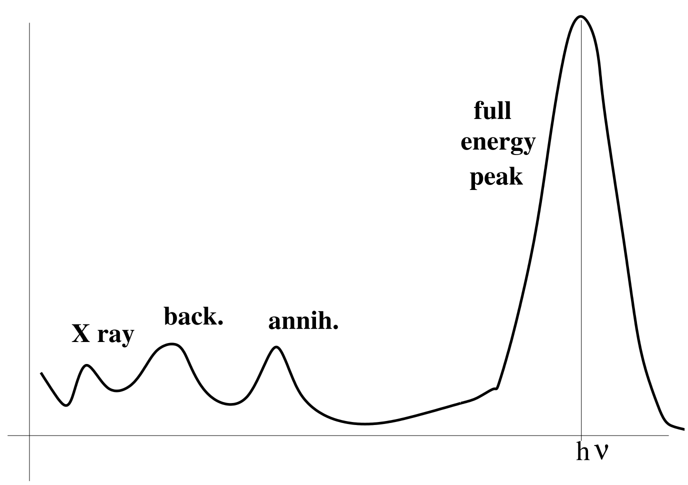
a, e photoelectric effect; b Compton effect;
c pair production; d backscattering;
f Compton edge ($E_{\rm max}$ see page 28)



Gamma spectrum in a big detector



All the processes release at the end **the primary γ energy**
The material surrounding the detector can give:
a **backscattering;** b **0.511 MeV annihilation γ;**
c **X ray from photoeffect in the screen;**



0.24 Charged Particles

The charged particles are:
e^+ e^- **p** α **ions (charged nuclei) nuclear fragments**

Historically, e^+ **and** e^- **are called** β **rays ad the** e^- **coming from the inner atomic shells are called** δ **rays**
The α **particles or** α **rays are simply the** ^4He **nucleus.**

All the charged particles in matter are subject to:

(1) continuous energy loss by ionization

$$\text{collision energy loss} \quad \left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)_{\text{coll}} \begin{cases} \text{MeV/cm} \\ \text{MeV/(cm}^2\text{ g)} \end{cases}$$

(2) continuous energy loss by radiation
(when $E \gg mc^2$**,** $\gamma \gg 1$**)**

$$\text{bremsstrhalung} \quad \left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)_{\text{rad}} \begin{cases} \text{MeV/cm} \\ \text{MeV/(cm}^2\text{ g)} \end{cases}$$

$$\text{Stopping power}: \left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)_{\text{coll}} + \left(\frac{\mathrm{d}E}{\mathrm{d}x}\right)_{\text{rad}}$$

(3) Coulomb collisions with nuclei (scattering)

$$\sigma_{\text{sc}} \quad \begin{cases} \text{barn} = 10^{-24}\text{ cm}^2 \\ \text{fm}^2 = 10^{-26}\text{ cm}^2 \end{cases}$$

0.25 Energy loss by collisions

Due to the long-range Coulomb force, the collisions with the electrons of the absorber atoms are so numerous that they appear as a continuous process.
A collision can give the atom ionization or excitation and these processes are used in the detectors of charged particles
The collision energy loss is well described by the Bethe-Bloch formula:

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{4\pi e^4 z^2 N Z}{m_e c^2 \beta^2}\left[\frac{1}{2}\ln\frac{2m_e c^2\beta^2\gamma^2 T_{\max}}{I^2} - \beta^2\right]$$
$$= 0.3071\,\rho\,\frac{z^2 Z}{A\beta^2}\left[\frac{1}{2}\ln\frac{2m_e c^2\beta^2\gamma^2 T_{\max}}{I^2} - \beta^2\right] \quad \left[\frac{\mathrm{MeV}}{\mathrm{cm}}\right]$$

z, Z **are the atomic numbers of the projectile and absorber atoms and**

$$4\pi e^4 N_A/(m_e c^2) = 0.3071 \;\; \mathrm{MeV\ cm^2/g} \tag{8}$$

m_e **and** M **are the electron and projectile mass (eV)**
$T_{\max}$ **is the max energy transferred to an electron**

$$T_{\max} = \frac{2m_e c^2\beta^2\gamma^2}{1 + 2\gamma m_e/M + (m_e/M)^2}\;, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{9}$$

$\beta = 1 - v^2/c^2$ **where** v **is the projectile velocity**
ρ **is the density and** I **is the ionization potential:**

$$I \simeq 12 \times Z \quad [eV]$$

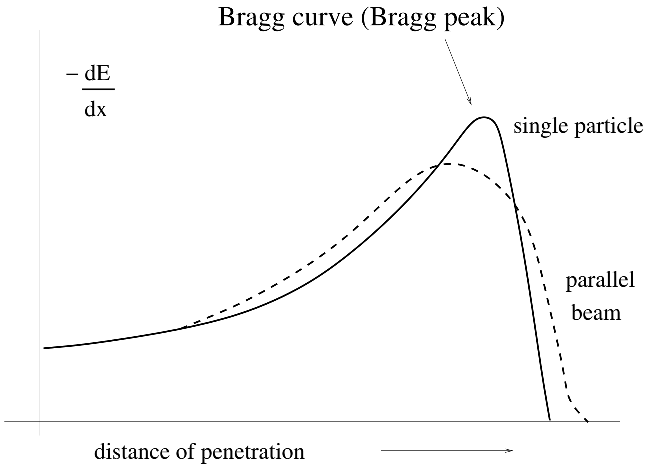
All the charged particles follows this formula!
Some minor corrections at very low and very high energies are necessary.
Often it is used also: $\frac{\mathrm{d}E}{\mathrm{d}(\rho\,x)}$ $\left[\frac{\mathrm{MeV\ cm^2}}{\mathrm{g}}\right]$

Energy loss by collisions

The collisional energy loss has a general behaviour as

$$\frac{\mathrm{d}E}{\mathrm{d}x} \propto \frac{1}{v^2}$$

that is more the particle is low more the $\mathrm{d}E/\mathrm{d}x$ is high



This behaviour is more and more evident by increasing the projectile mass and, at the same mass, for antiparticles (Barkas effect)

Example: 1 MeV Electron on an Al absorber

$$E = \frac{m}{\sqrt{1-\beta^2}} = (1+0.511) \rightarrow \beta = \sqrt{1-m^2/E^2} = 0.94$$

$z = 1,\ Z = 13,\ A = 27,\ \rho = 2.7$ **g/cm^3,** $\gamma = 2.93$

$T_{\max} = 0.987$ **MeV,** $I = 12 \times 13 = 156$ eV $= 156\ 10^{-6}$ MeV

$$\frac{\mathrm{d}E}{\mathrm{d}\rho\, x} = 1.480 \ \ \mathrm{MeV\ cm^2/g} = 4.00 \ \ \mathrm{MeV/cm}$$

(The more precise result with the density correction is 1.473 MeV cm^2/g.)

Mixtures and compounds

From (6) and pages 20 and 24:

$$\frac{\mathrm{d}E}{\mathrm{d}(\rho\, x)} = \sum_i w_i \left[\frac{\mathrm{d}E}{\mathrm{d}(\rho\, x)}\right]_i \quad \left[\frac{\mathrm{MeV\ cm}^2}{\mathrm{g}}\right]$$

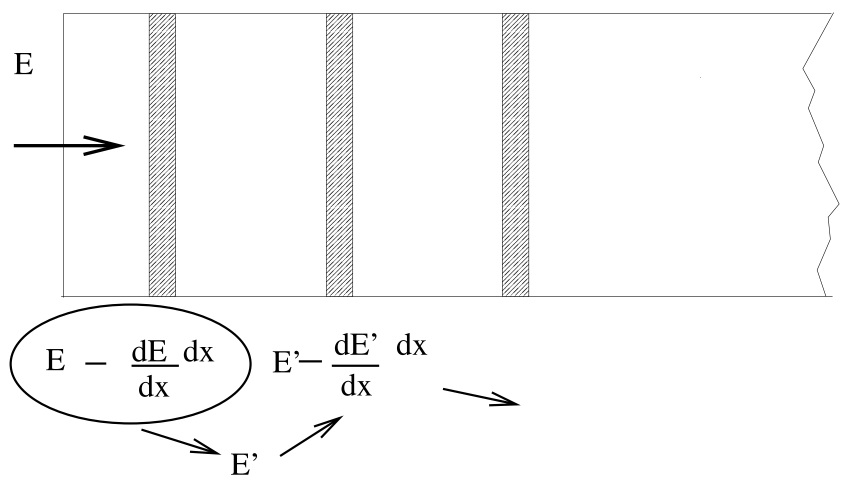
When the total energy loss is calculated the thickness must be expressed as the transparency.

$$\Delta E = \frac{\mathrm{d}E}{\mathrm{d}(\rho\, x)}\ \rho\ x$$

Range

$$R = \int_E^0 \frac{\mathrm{d}x}{\mathrm{d}E}\ \mathrm{d}E$$

This integral must be done carefully or solved with a simulation



More and more thin layers are added until the energy is zero.

The total path is the range

0.26 Radiation Energy Loss

According to Maxwell theory an accelerated (decelerated) charge loses energy by photon emission.
This radiation is called synchrotron radiation (from circular orbits) or bremsstrahlung (motion in matter) when a fast ($\gamma \gg 1$) charged particle decelerates in the field of a nucleus partially screened by the atomic electrons.
This is as an X-ray machine works.
Useful formulae for energy loss calculation (MeV/cm):

$$-\left[\frac{\mathrm{d}E}{\mathrm{d}x}\right]_{\mathrm{rad}} = \frac{0.3071E\ Z(Z+1)}{4\,\pi\,m_e\,c^2\,137}\frac{\rho}{A}\left[4\ln\frac{2E}{m_e c^2} - \frac{4}{3}\right],\quad E < 137\,m_e c^2\,Z^{-1/3}$$

$$-\left[\frac{\mathrm{d}E}{\mathrm{d}x}\right]_{\mathrm{rad}} = \frac{0.3071E\ Z(Z+1)}{4\,\pi\,m_e\,c^2\,137}\frac{\rho}{A}\left[4\ln(183Z^{-1/3})\right],\quad E \gg 137\,m_e c^2\,Z^{-1/3}$$

they are accurate within $10 \div 20\%$ with the standard tables. Note the asymptotic behaviour as $\simeq EZ^2$

The mean angle for photon emission is

$$\langle\theta_\gamma\rangle \simeq \frac{m_e c^2}{E}$$

Most of radiation lies inside a narrow cone along the incident charged particle direction. The cone is more and more narrow with increasing the energy.

Example: electrons on Al nuclei with 1, 10, 100 MeV:
$E_1 = 1.511,\ E_2 = 10.511,\ E_3 = 100.511$ **MeV,**
Energy loss at the three energies:
$\mathrm{d}E/\mathrm{d}(\rho x) = 0.0206,\ 0.335,\ 4.09$ **MeV cm^2/g**
Accurate table values: $0.029,\ 0.287,\ 3.71$ **MeV cm^2/g.**

Radiation length

At high energy the bremsstrahlung follows the rule (remember (8) at page 40):

$$-\left[\frac{\mathrm{d}E}{\mathrm{d}x}\right]_{\mathrm{rad}} = \frac{0.3071\ \ Z(Z+1)}{4\,\pi\,m_e\,c^2\,137}\frac{\rho}{A}\,[4\ln(183Z^{-1/3})]E \equiv \frac{1}{X_0}\,E$$

which implies an energy loss of the type

$$E = E_0\mathrm{e}^{-x/X_0} \qquad (10)$$

where

$$X_0 = \frac{4\,\pi\,m_e\,c^2\,137}{0.3071\ \ Z(Z+1)}A\,\frac{1}{4\ln(183Z^{-1/3})} \quad \mathrm{g/cm^2}$$

There is the more accurate empirical formula of Dahl (data interpolation):

$$X_0 = \frac{716.4\ \ A}{Z(Z+1)\ \ln(287/\sqrt{Z})}\ , \quad \frac{\mathrm{g}}{\mathrm{cm^2}}$$

The radiation length X_0 (sometimes denoted as X_R) is the mean distance over which a high energy particle (electron) remains with a fraction $1/e \simeq 37\%$ of its initial energy, the remainder being lost by bremsstrahlung.

The radiation length is the characteristic distance for describing the electromagnetic cascades.

Radiation Length

Since the radiation energy loss depends on the atomic number as $1/A$, for a mixture or compound the usual rule follows (see page 20)

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \sum_i w_i \left[\frac{\mathrm{d}E}{\mathrm{d}x}\right]_i$$

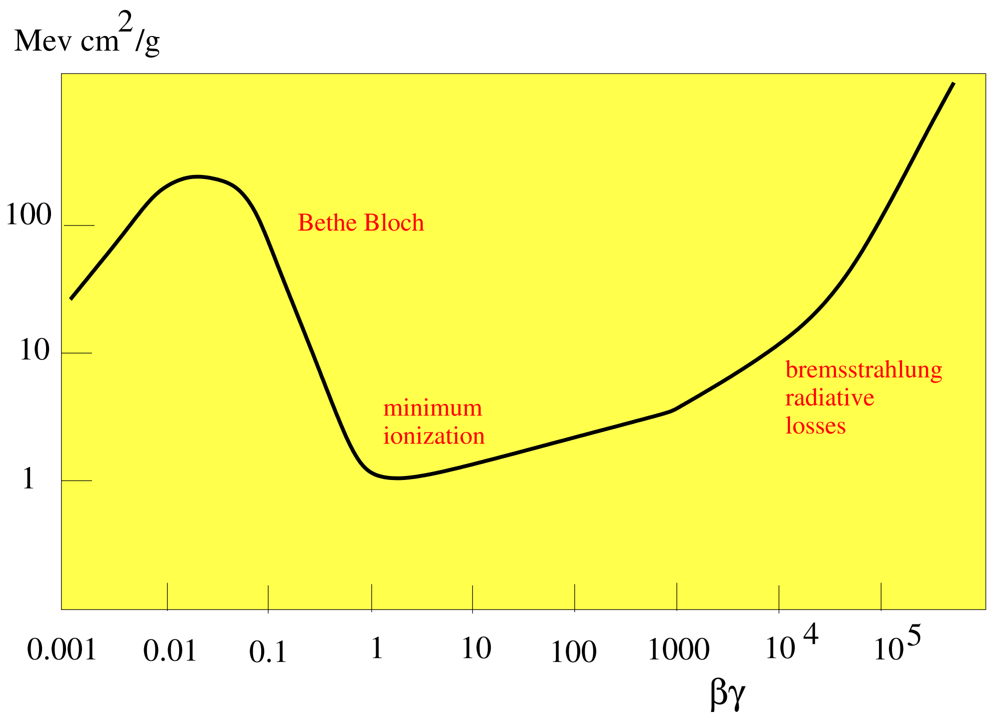
From this an approximate rule follows also for the radiation length:

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \sum_i w_i \left[\frac{\mathrm{d}E}{\mathrm{d}x}\right]_i = E \sum_i \frac{w_i}{X_i} = \frac{E}{X_0}$$

$$\frac{1}{X_0} = \sum_i \frac{w_i}{X_i}$$

0.27 Stopping Power

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \left[\frac{\mathrm{d}E}{\mathrm{d}x}\right]_{\mathrm{coll}} + \left[\frac{\mathrm{d}E}{\mathrm{d}x}\right]_{\mathrm{rad}}$$



All the incident particles have a region of minimum ionization. MIP: minimum ionizing particle:

$$\left[\frac{\mathrm{d}E}{\mathrm{d}(\rho\, x)}\right]_{\mathrm{MIP}} \simeq 2 \;\; \frac{\mathrm{MeV\;\; cm}^2}{\mathrm{g}}$$

for $\beta\gamma \simeq 3$**.**

The general rule for $e^+\; e^-$ **collision/radiation balance:**

$$\frac{(\,\mathrm{d}E/\,\mathrm{d}x)_{\mathrm{rad}}}{(\,\mathrm{d}E/\,\mathrm{d}x)_{\mathrm{coll}}} \simeq \frac{EZ}{1600\, m_e c^2} \simeq \frac{EZ}{800} \;\rightarrow\; E_{\mathrm{crit}}(\mathrm{MeV}) = \frac{800}{Z} \qquad (11)$$

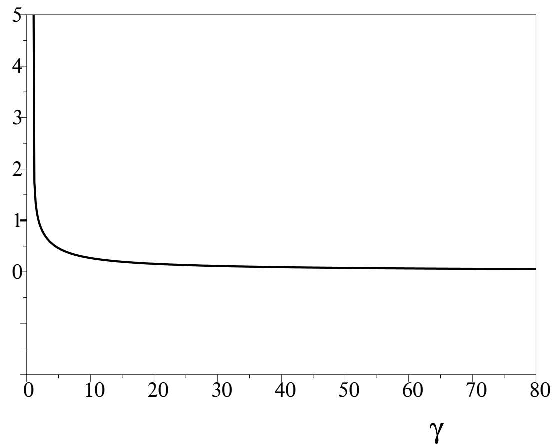
0.28 Positronium annihilation

$$e^+ + e^- = \gamma + \gamma$$

Annihilation into a single photon is possible with an electron bound in a nucleus, but the cross section is much lower ($< 20\%$). The cross section is:

$$\sigma_{\rm ann} = \pi \frac{e^4}{m_e^2\, c^4} \frac{1}{\gamma_e + 1} \left[\frac{\gamma_e^2 + 3\gamma_e + 1}{\gamma_e^2 - 1} \ln(\gamma_e + \sqrt{\gamma_e^2 - 1}) - \frac{\gamma_e + 3}{\sqrt{\gamma_e^2 - 1}} \right]$$

where $\gamma_e = E/m_e c^2$. The cross section peaks for $\gamma = 1$,



that is for positrons at rest, where the $e^+\ e^-$ system can form the positronium:
singlet $e^+\ e^- \rightarrow 2\,\gamma$, 0.511 MeV each, lifetime 0.1 ns;
triplet $e^+\ e^- \rightarrow 3\,\gamma$, lifetime 100 ns;
Triplet:singlet is 3:1, but in dense media, due to the longer lifetime, the triplet undergoes many collisions that favour the transition to singlet and the sudden decay into 2 γ (2γ dominance).

0.29 Energy straggling (dispersion)

The stopping power in a thickness X of absorber is the MEAN VALUE of a statistical process
For the Central Limit theorem for thick absorbers ($\Delta E/E >$ 10%) the distribution is Gaussian.
For thin absorbers the distribution is strongly asymmetrical with a long right tail in the lost energy (Landau and Vavilov). The kind of the distribution is decided by some scale parameters:
the maximum energy transfer to an electron (page 40)

$$E_{\max} = \frac{2m_e c^2 \beta^2 \gamma^2}{1 + 2\gamma m_e/M + (m_e/M)^2}$$

the typical mean energy loss (page 40)

$$\xi = \frac{0.3071}{2} \frac{z^2 Z}{\beta^2} \frac{\rho}{A} X \quad \text{MeV}$$

the variance of the distribution

$$\sigma_E^2 = \xi \, E_{\max} \left(1 - \frac{\beta^2}{2}\right) \quad \text{MeV}^2 \qquad (12)$$

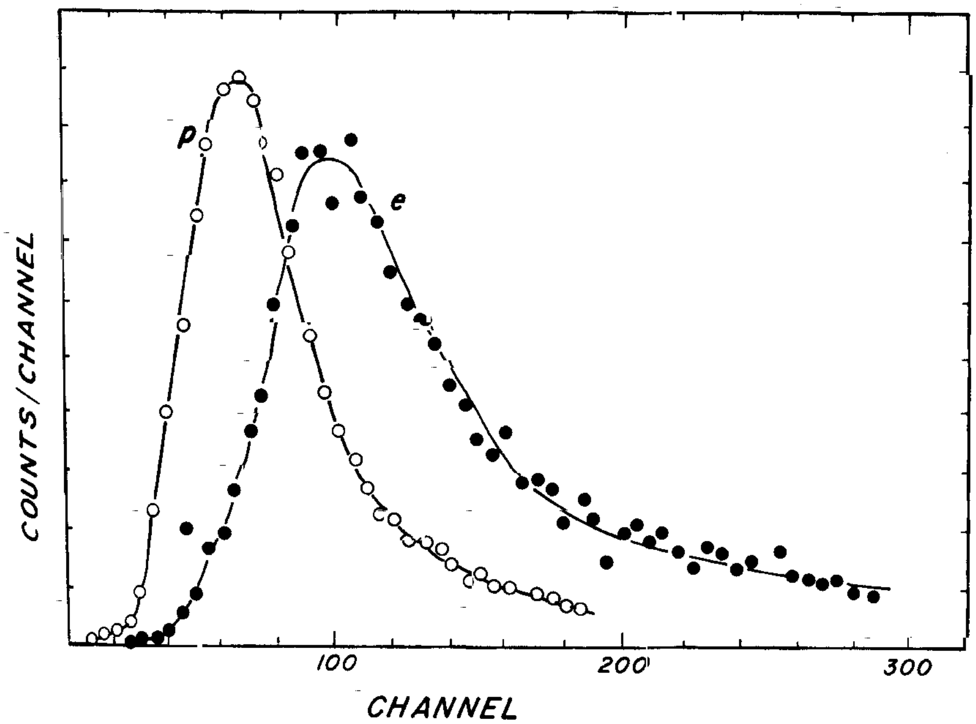
- $\xi/E_{\max} \ll 1$: **several collisions: Landau distribution**
- $\xi/E_{\max} \simeq 1$: **many collisions: Vavilov distribution**
- $\xi/E_{\max} \gg 1$: **great number of collisions, stochastic regime, Gauss distribution**

This theory assumes that $\xi/I \gg 1$, that is it neglects the fluctuations in the small energy losses, and considers only those due to δ electrons.
For $\xi/I < 1$ there is no solution (MC simulations)

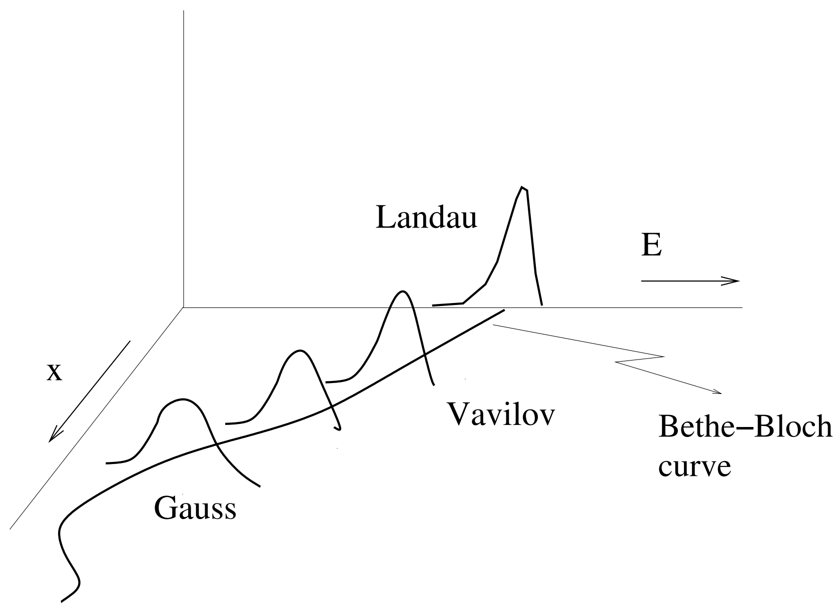
Landau-type curves

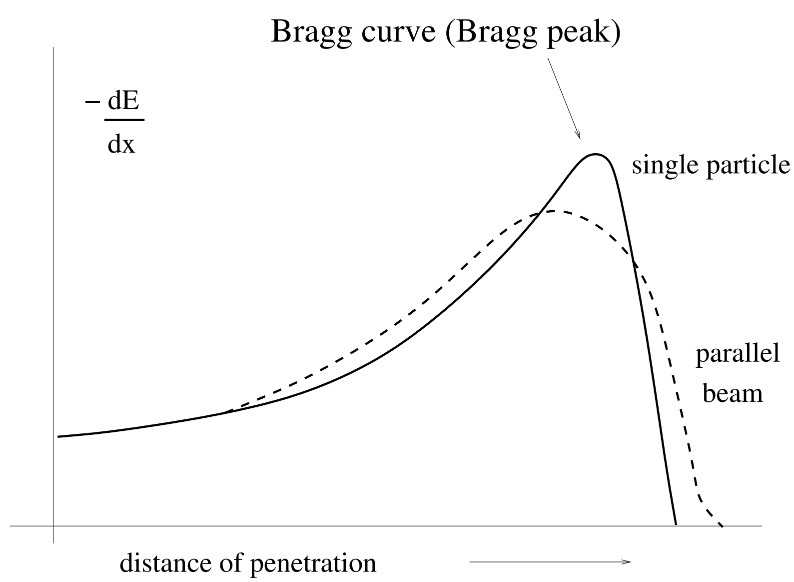
The Landau curve is the limit distribution of the theory for thin absorbers: it is an universal curve both for heavy particles and electrons
The detectors give the Landau curve as a function of the lost energy
The left tail is $\simeq 1.5\ \sigma$
The right tail extends up to $\simeq 9\ \sigma$ and it is due to the δ electron emission.

Figure 2.7 Measured pulse height distributions for 3-GeV/*c* protons and 2-GeV/*c* electrons in a 90% Ar + 10% CH_4 gas mixture. (After A. Walenta, J. Fischer, H. Okuno, and C. Wang, Nuc. Instr. Meth. 161: 45, 1979.)

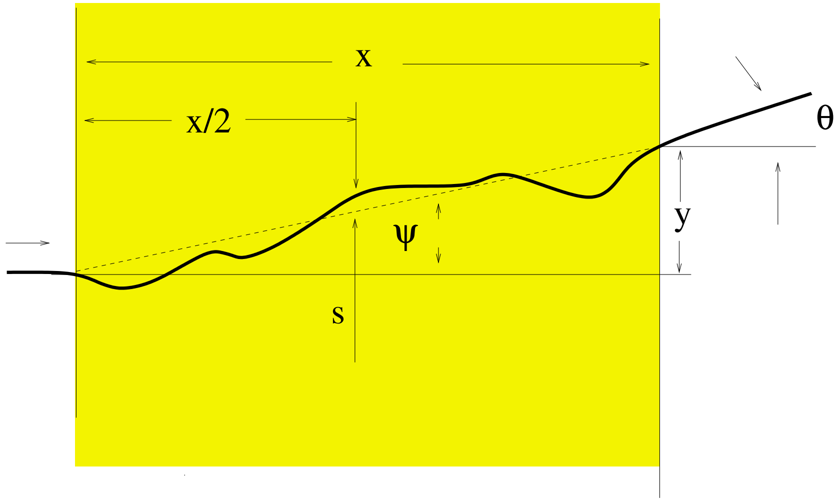


Energy and range straggling





0.30 Coulomb Multiple Scattering



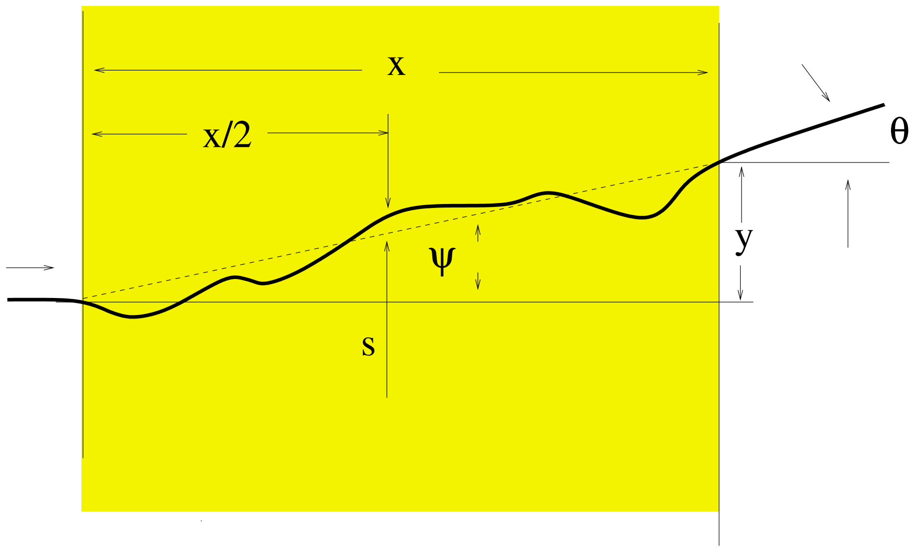
The charged particle traversing a medium experiences the effect of the screened Coulomb field of the nuclei. Since the elastic scattering cross section behaviour $\simeq 1/\sin^4(\theta/2)$, at small angles the effect is so high that it can be treated as a continuous process giving small angle deflections per unit path.

The full treatment is given by the Molière theory. An empirical formula deduced from it gives the r.m.s. deflection angle with an accuracy $\simeq 10\%$:

$$\theta_0 = \frac{13.6\ \mathrm{MeV}}{\beta c p}\, z \sqrt{x/X_0}\,[1 + 0.038 \ln(x/X_0)] \quad \mathrm{rad} \qquad (13)$$

CAUTION: since this is an empirical formula with a logarithm, if one adds two thin media the resulting r.m.s. angle is not $\sqrt{\theta_{01}^2 + \theta_{02}^2}$. The rule is to calculate before x and X_0 (in cm^2/g, as the usual weighted sum) and after to use the formula.

Coulomb Multiple Scattering



The Molière distribution, in the small angle approximation, in the plane can be approximated with a gaussian: indexdistribution!Gauss

$$\frac{1}{\sqrt{2\pi}\,\theta_0}\ \exp\left[-\frac{\theta^2}{\theta_0^2}\right]\ \ d\theta \tag{14}$$

In space the distribution with gaussian component is given by the Rayleigh distribution:

$$\frac{1}{2\pi\,\theta_0^2}\ \exp\left[-\frac{\theta_x^2+\theta_y^2}{\theta_0^2}\right]\ \ d\theta_x\, d\theta_x \tag{15}$$

where x and y are in the plane $\perp$ to the direction of motion. In the small angle approximation:

$$\psi = \frac{1}{\sqrt{3}}\theta_0\ , \quad y = \frac{1}{\sqrt{3}}\,x\,\theta_0\ , \quad s = \frac{1}{4\sqrt{3}}\,x\,\theta_0$$

Coulomb Multiple Scattering: electrons

Il passaggio delle radiazioni nella materia 39

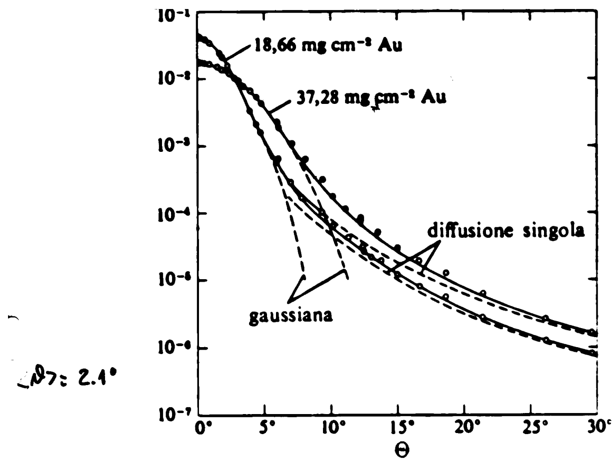


Figura 2.15. Distribuzione angolare di elettroni di 15.7 MeV diffusi da Au. Le curve continue indicano la distribuzione prevista dalla teoria di Molière della diffusione multipla a piccoli e a grandi angoli, con una estrapolazione nella regione di transizione: le curve tratteggiate, le distribuzioni secondo la teoria gaussiana e della diffusione singola. L'ordinata dà il logaritmo della frazione di fascio diffuso entro 9.696×10^{-3} sr. [R. D. Birkhoff in (Fl E)].

Electrons and heavy particles have more or less the same formula for the $\mathrm{d}E/\,\mathrm{d}x$ and the multiple scattering. However, the real behaviour, for energies around the MeV, is completely different:

- the electron are very often relativistic and the energy loss has a large bremsstrahlung component;
- the energy loss for heavy particles is mainly due to excitation/ionization (Bethe-Bloch);
- the electrons have large multiple scattering deviations and their motion into a medium is “zigzagged” (see the next photo)

Coulomb Multiple Scattering: electrons

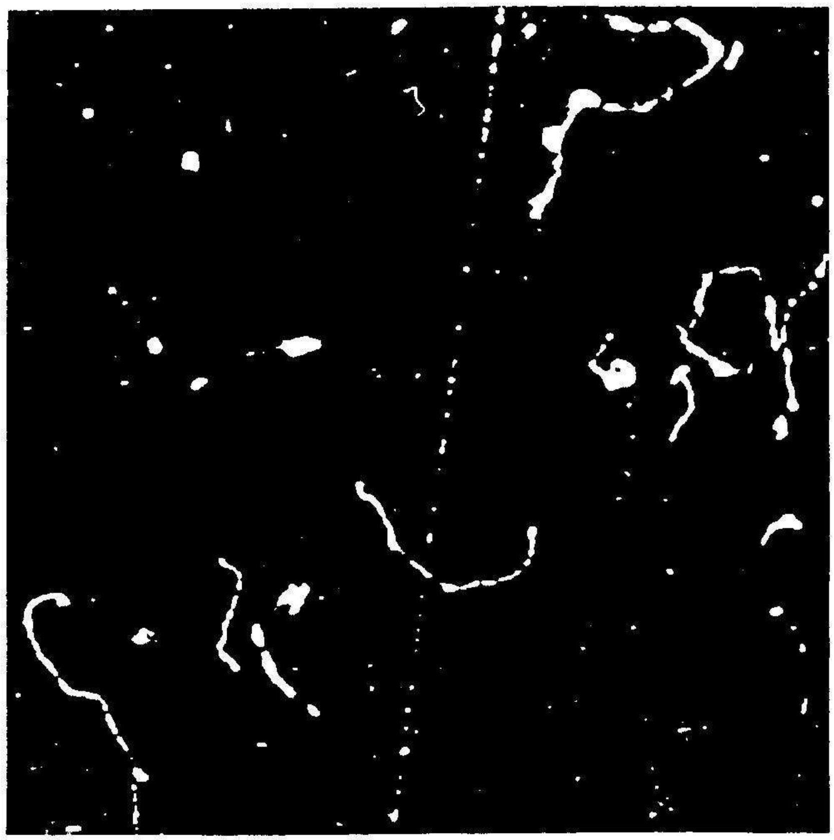


Figura 2.11. Elettroni lenti che presentano un cammino incurvato a causa della diffusione. Un elettrone veloce procede in linea retta. [Foto originale di Wilson, 1923].

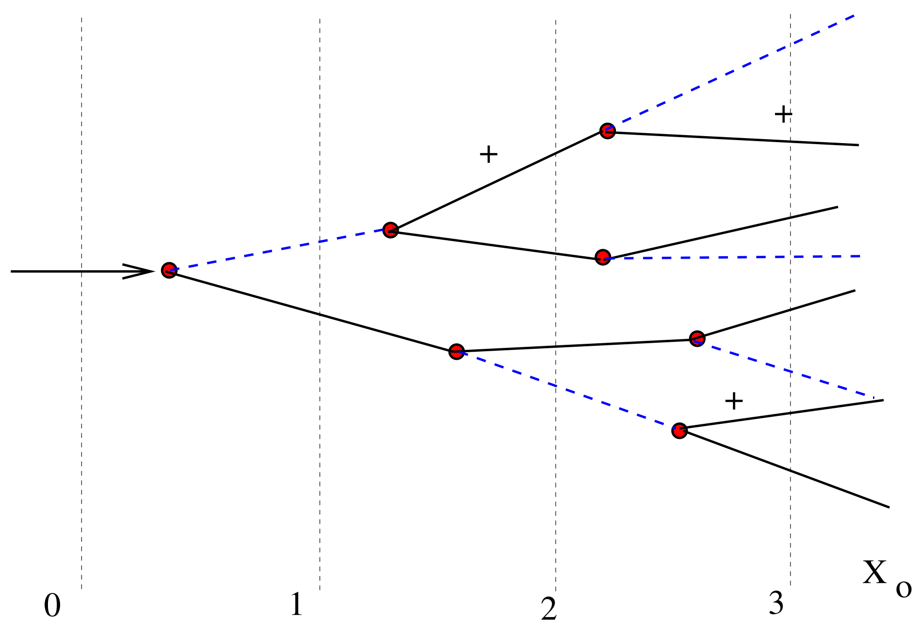
0.31 Electromagnetic showers

Prima osservazione di uno sciame da parte di Blackett e Occhialini in un

Electromagnetic (e.m.) showers

High energy electrons radiates high energy photons and lose energy exponentially in a radiation length X_0. (see page 44)
High energy photons generate high energy electrons by pair production. The mean distance is $(7/9)\,X_0$.

These two combined effects are the source of the spectacular e.m. showers.



The two important quantities are:
the distance measured in radiation lengths: $t = x/X_0$
the critical energy below which
$(\,\mathrm{d}E/\,\mathrm{d}x)_{\mathrm{rad}} < (\,\mathrm{d}E/\,\mathrm{d}x)_{\mathrm{coll}}$ **(page 46)**

Electromagnetic showers

The structure of an e.m. shower triggered by a particle (electron or photon) with energy E_0 is:

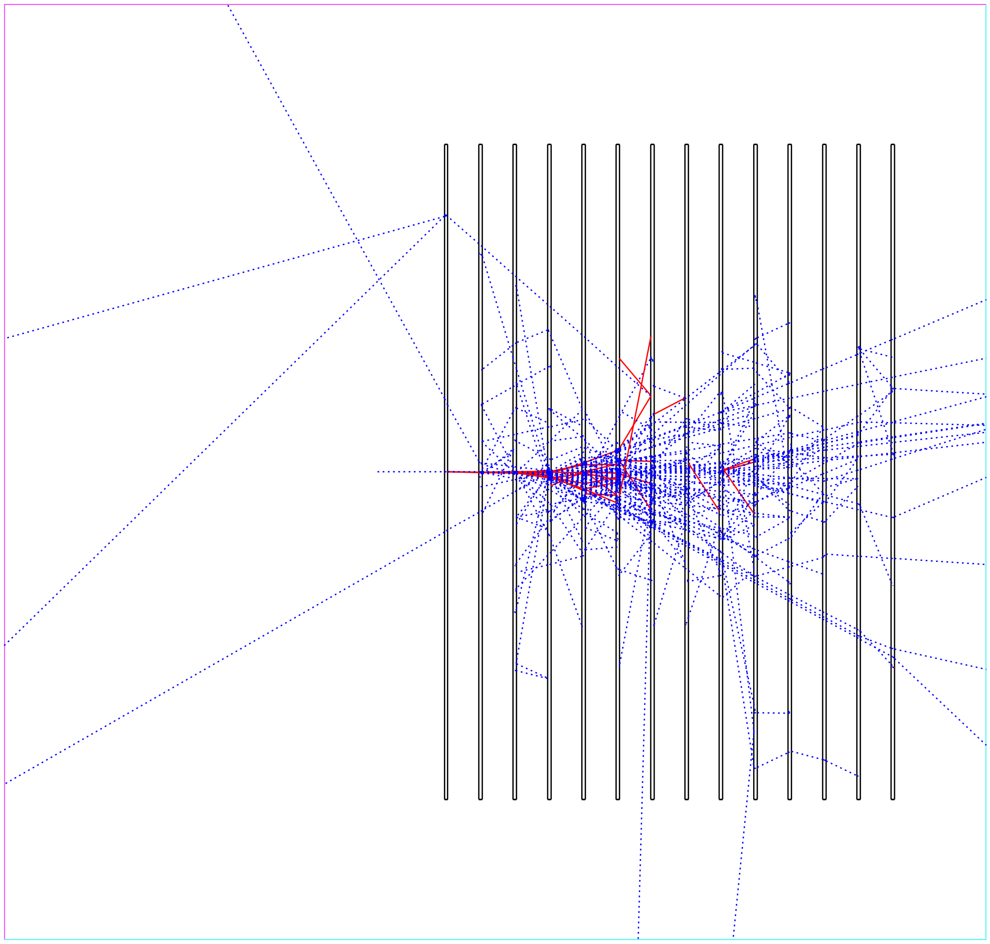
- **number of particles after t radiation lengths** $N(t) \simeq 2^t$
- **distance with shower energy E_t:** $t(E_t) = \ln(E_0/E_t)/\ln 2$
- **distance with the maximum number of particles. This roughly is the shower depth, because after this point the shower abruptly stops.**

$$t_{\max} = \frac{\ln E_0/E_c}{\ln 2}$$

  We see that the shower depth increases logarithmically with the primary energy.
- **the mean number of particles (e^+, e^-, γ) is $N_{\max} = E_0/E_c$ and is proportional to the primary energy.**

e.m. showers occur in "normal life" by cosmic rays and artificially in the particle accelerators.

A 1 Gev γ shower

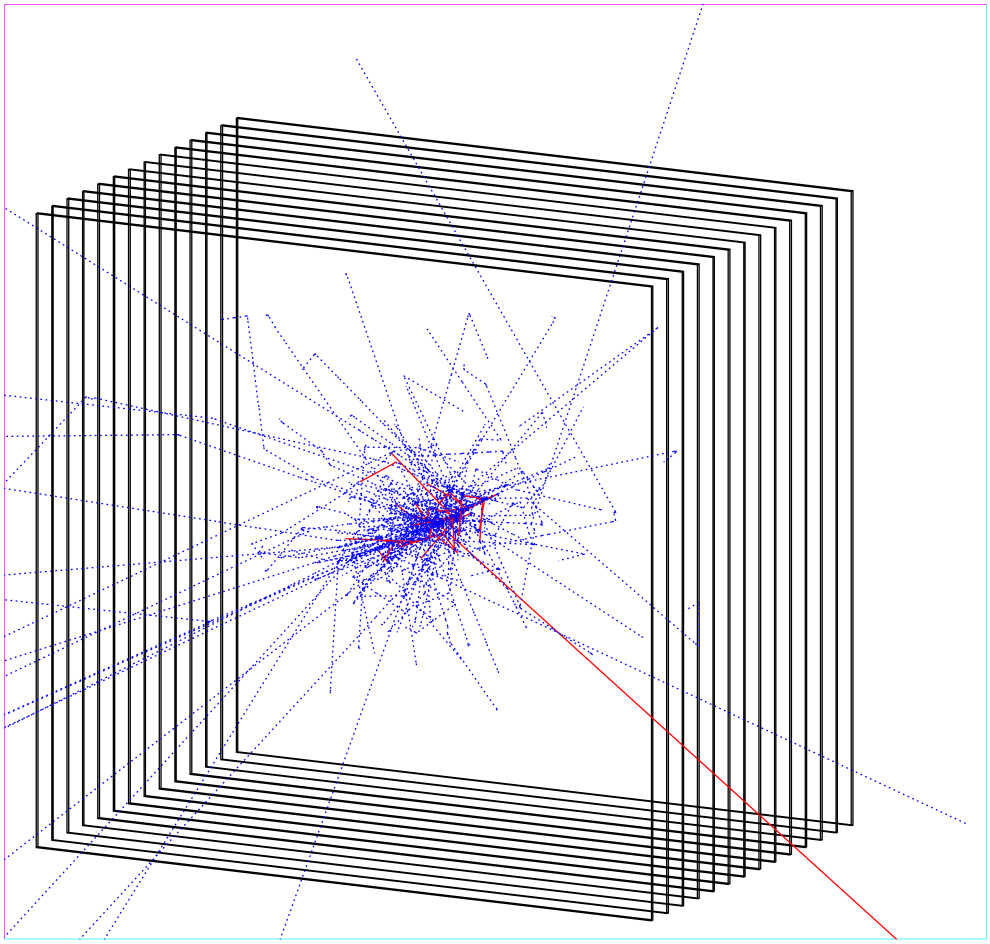


$< -1\text{m} - - >$

γ rays, charged particles

14 Copper slabs 1 cm thick, $X_0 = 1.43$ cm,
$E_c = 27$ MeV, $t_{\max} = 5$, $x_{\max} = 7.45$ cm, $\simeq 8$ slabs

A 1 Gev γ shower



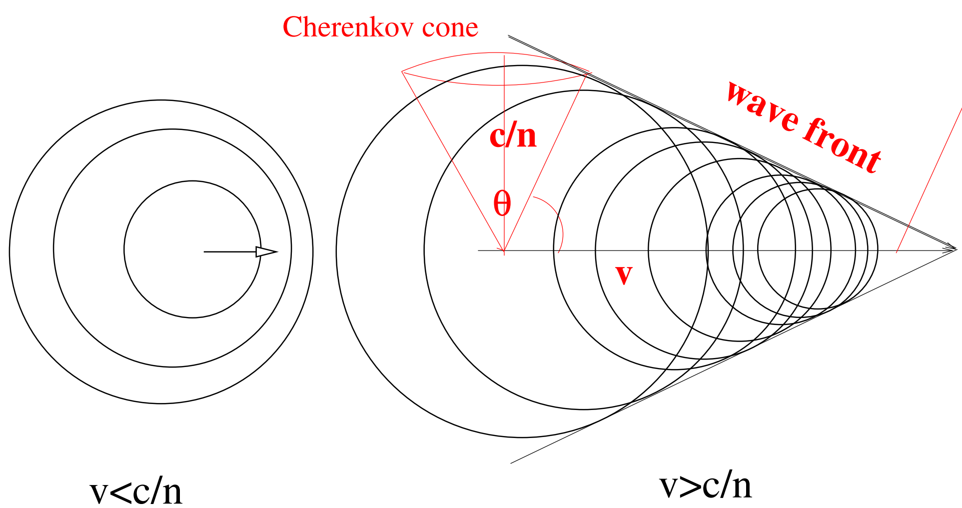
2 m × 2 m calorimeter
γ rays, charged particles

0.32 Cherenkov radiation

A particle with constant velocity does not radiate. However, the electrons of the medium feel a variable e.m. field, they accelerate/decelerate and emit a small amount of radiation. This effect is a negligible contribution to the particle energy loss.
However, when the particle velocity exceeds the light velocity in a medium of refractive index n

$$v > \frac{c}{n}\ , \quad \rightarrow \quad \beta > \frac{1}{n}$$

the coherent wavefront of the Cherenkov light can be detected (think to a fast ship in water...)



$$\cos\theta_c = \frac{c}{n\ v} = \frac{1}{n\ \beta}$$

The number of γ per unit path length and per energy interval is

$$\frac{\mathrm{d}^2 N}{\mathrm{d}E\ \mathrm{d}x} = \frac{\alpha z^2}{\hbar\ c}\sin^2\theta_c \simeq 370\, z^2\, \sin^2\theta_c \quad \mathrm{eV}^{-1}\ \mathrm{cm}^{-1}$$

A Cherenkov detector

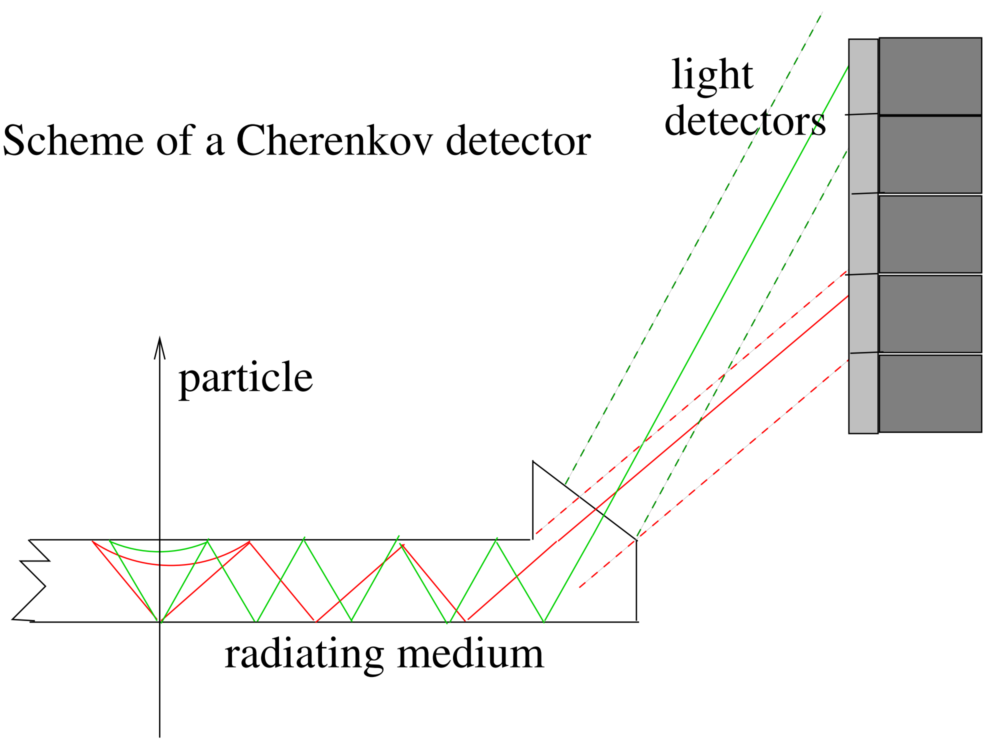
Since

$$\cos\theta_c = \frac{c}{n\ v} = \frac{1}{n\ \beta}$$

a measurement of the Cherenkov angle permits to measure the β of the particle when $\beta > 1/n$.

Once β is known, a measurement of the particle momentum (i.e. track curvature in a magnetic field) allows the determination of the particle mass

Scheme of a Cherenkov detector



0.33 Nuclear Interactions

Differently from the e.m. interactions, the nuclear interactions due to the short range strong force give rise to discrete processes.

The e.m. interactions are forward peaked. Away from the forward direction, the main effects are due to the strong (nuclear) interaction.

Remember the connection between event rate and the cross section (page 18):

$$\frac{\#\ \text{collisions}}{\text{s}} = \sigma ISXN = \sigma ISX\frac{\rho N_A}{A} \quad (16)$$

Cross sections are related to nuclear radius. In a blob of constant density one has $(4/3)\pi r^3 \propto A$, therefore;

$$r = r_0 A^{1/3} \quad \text{where} \quad r_0 \simeq 1.25\ 10^{-13}\ \text{cm}$$

Nuclear Interactions

The general behaviour of the cross section is

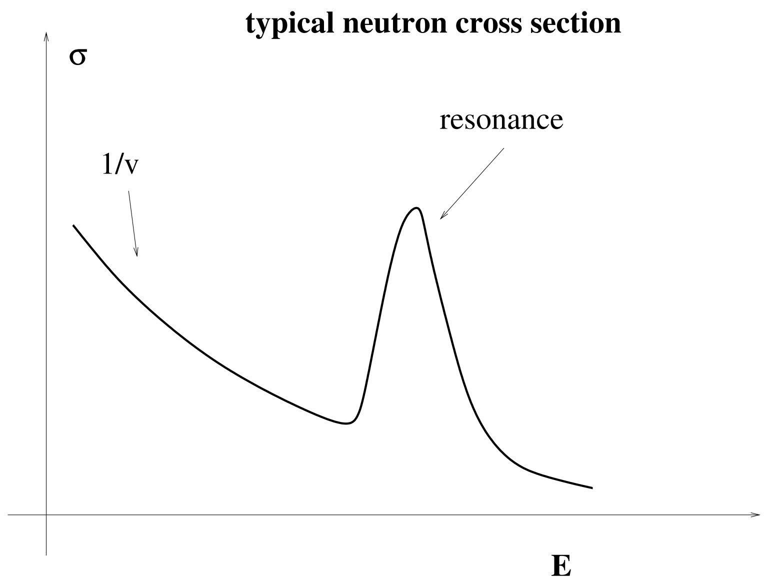
$$\sigma_t = \sigma_{\text{el}} + \sigma_{\text{abs}}$$

From Quantum Mechanics, the scattering cross section at low energy is 4 times the geometrical cross section:

$$\sigma_{\text{el}} = 4\,\pi\,r^2 = 4\,\pi\,r_0^2\,A^{2/3}$$

The capture cross section at low energy has the $1/v$ behaviour

$$\sigma_{\text{abs}} = \frac{\Gamma}{(E - E_0)^2 + \Gamma^2/4} \quad \to \frac{c}{\sqrt{E}}$$



$$\sigma_t = 4\,\pi\,r^2 + \frac{c}{\sqrt{E}}$$

The 1/v behaviour

F is the number of collisions (absorptions)/cm^{3}s:

$$F = \int n(E)\, v(E)\, \Sigma(E)\, \mathrm{d}E \ , \tag{17}$$

in the case of $1/v$ behaviour:

$$\Sigma(E)\, v(E) = \Sigma_0\, v_0 = \text{constant} \ ,$$

where v_0 is a point chosen in the range of the $1/v$ behaviour. Conventionally the point is chosen at room temperature (20^0 C):

$$v_0 = 2\,200 \ \text{m/s} \quad \text{corresponding to} E = 0.0253 \ \text{eV}$$

From the previous equations one has:

$$F = \Sigma_0 v_0 \int n(E)\, \mathrm{d}E = \Sigma_0 v_0\, n = \Sigma_0 \phi_0 \tag{18}$$

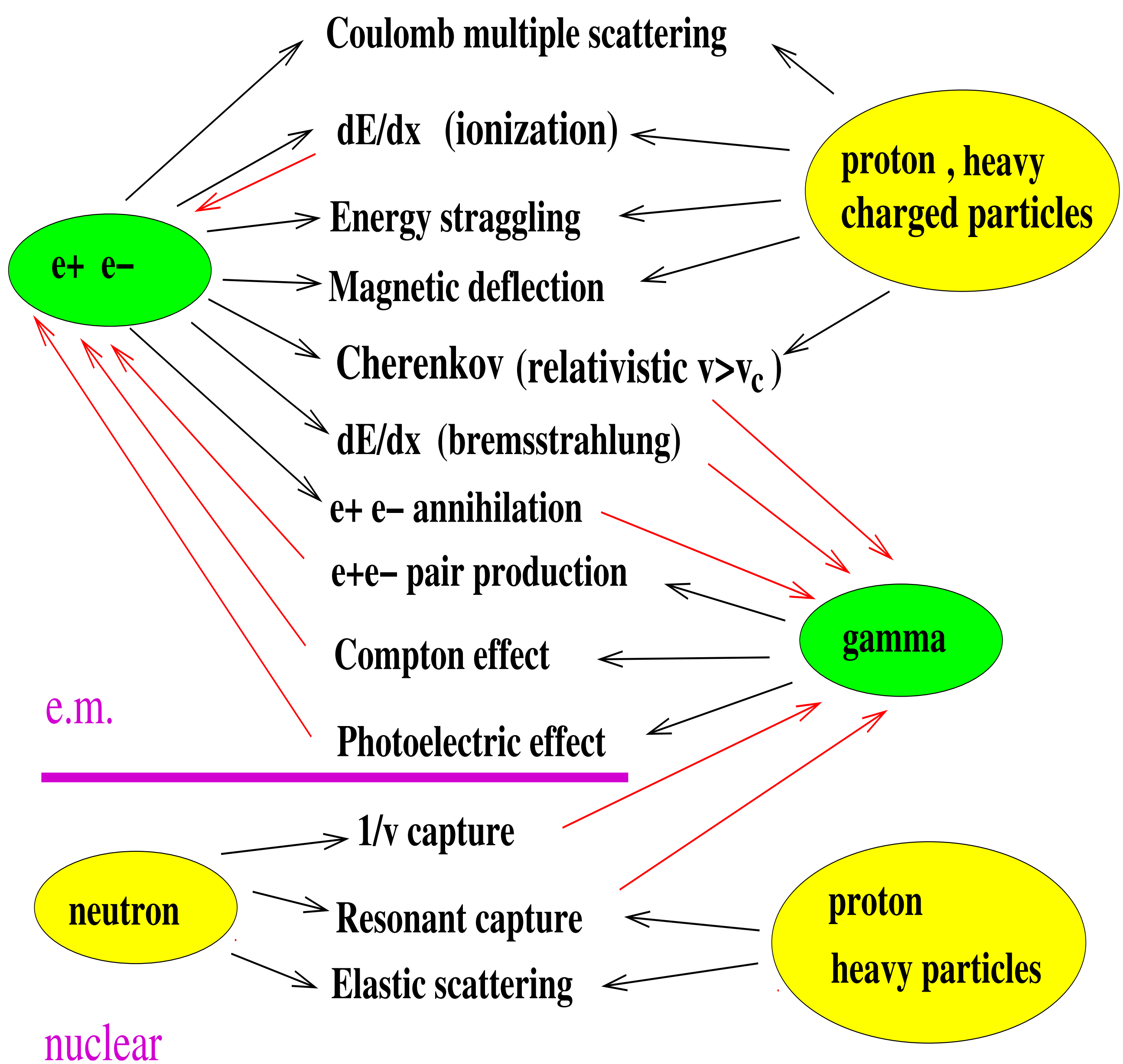
and the collision rate is calculated as for a monoenergetic beam at room temperature. Small deviations from the $1/v$ behaviour are considered through an empirical factor g:

$$F = g\, \Sigma_0\, \phi_0 \tag{19}$$

Some values at $v_0 = 2\,200$ m/s:

symbol	Cd	In	U	Pu
g	1.32	1.02	0.97	1.00
σ_a (barns)	2450.	193.5	7.6	1011.
Σ_a (cm^{-1})	113.5	7.42	0.37	49.9

0.34 Interaction matter-radiation: summary



0.35 Example: 10 MeV protons

Consider a 0.01 cm (100μ) thick Al absorber. Calculate the energy loss, straggling and mean multiple scattering deviation for 10 MeV protons.
Calculate the nuclear interaction probability when $\sigma = 1$ barn.
AL: Z=13, A=27, $X_0 = 8.9$ cm, $\rho = 2.7$ g/cm^3

	proton	electron
β	0.1448	0.9988
γ	1.0106	20.57
$E_{\max}$	0.0219 MeV	10. MeV
p	137.35 MeV/c	10.49 MeV/c
ξ	0.0952 MeV	0.002 MeV

10 MeV Protons

from page 40 $\mathrm{d}E/\mathrm{d}x = 93.7$ MeV/cm or $\mathrm{d}E/\mathrm{d}x = 34.71$ MeV cm^2/g, hence the energy loss is
$\Delta E = 93.7 \times 0.01 = 0.937$ **MeV.**
From page 48 we have $\sigma = 0.045$ and $\xi/E\mathrm{max} = 4.34$ and the distribution is $\simeq$ gaussian.

$$\Delta E = 0.937 \pm 0.045 \quad \mathrm{MeV}$$

From page 51 one has $\langle\theta\rangle = 0.0170$ radiants, $\langle\theta\rangle = 0.97^o$
Nuclear interaction probability:

$$P = \sigma\, \rho\, X \frac{N_A}{A} = 6\ 10^{-4}$$

0.36 Example: 10 MeV electrons

Consider a 0.01 cm (100μ) thick Al absorber. Calculate the energy loss, straggling and mean multiple scattering deviation for 10 MeV electrons.
AL: Z=13, A=27, $X_0 = 8.9$ **cm,** $\rho = 2.7$ g/cm^3

	proton	electron
β	0.1448	0.9988
γ	1.0106	20.57
E_{max}	0.0219 MeV	10. MeV
p	137.35 MeV/c	10.49 MeV/c
ξ	0.0952 MeV	0.002 MeV

10 MeV Electrons
from page 40 $\mathrm{d}E/\,\mathrm{d}x = 4.78$ **MeV/cm or** $\mathrm{d}E/\,\mathrm{d}x = 1.77$ **MeV cm^2/g, hence the collision energy loss is**
$\Delta E = 4.78 \times 0.01 = 0.0478$ **MeV.**
From page 43 $\mathrm{d}E/\,\mathrm{d}x = 0.640$ **MeV/cm by bremsstrahlung and** $\Delta E = 0.640 \times 0.01 = 0.0064$ **MeV.**
Total energy loss $\Delta E = 0.0478 + 0.0064 = 0.0542$ **MeV.**
From page 48 we have $\sigma = 0.10$ **and** $\xi/E\text{max} \ll 1$ **so that the distribution is Landau-type (note the large fluctuations):**

$$\Delta E = 0.054 \pm 0.100 \quad \text{MeV}$$

From page 51 one has $\langle\theta\rangle = 0.0322$ **radiants,** $\langle\theta\rangle = 1.85^o$

0.37 Neutron interactions

Neutron sources from (α, n) and (γ, n) reactions: Ra-Be, Po-Be
From accelerators or accelerating devices we use:

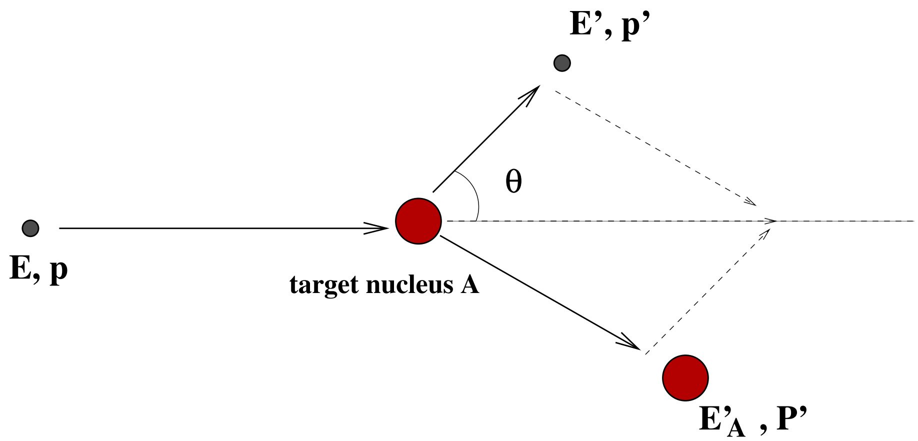
$$
\begin{aligned}
{}^7Li +^1 H &\rightarrow {}^7Be + n - 1.647 \text{ MeV} \\
{}^3H +^2 H &\rightarrow {}^4He + n + 17.6 \text{ MeV} \\
{}^2H +^2 H &\rightarrow {}^3He + n + 3.27 \text{ MeV} \ ,
\end{aligned}
$$

two-body reactions that give monoenergetic neutrons in CM but not in the LAB system.
Neutron-nucleus (neutron-matter) reactions:

- **elastic diffusion (moderation)**
- **inelastic reactions (moderation, activation)**
- **neutron absorption (n, γ), (n, α) (absorption, activation)**
- **fission (fuel)**

Moderators: Al, C, H_2O, D_2O (inelastic threshold are above 1 MeV or more)
Absorbers: Cd, Cs and many heavy nuclei, where absorption prevails over elastic scattering moderation

Diffusion in the LAB system



$$
\begin{aligned}
E &= E' + E'_A \quad \boldsymbol{p} = \boldsymbol{p}' + \boldsymbol{P}' \ , \quad p^2 = 2mE && (20)\\
P'^2 &= p^2 + p'^2 - 2pp' \cos\theta_L \\
V_2'^2 &= V_1^2 + V_1'^2 - 2V_1V_1' \cos\theta_L && (21)\\
ME'_A &= mE + mE' - 2m\sqrt{EE'}\cos\theta_L \ , \quad \frac{M}{m} \simeq A \\
AE'_A &= E + E' - 2\sqrt{EE'}\cos\theta_L \ , \quad E'_A = E - E' \\
E' &= \frac{E}{(A+1)^2}\left[\cos\theta_L + \sqrt{A^2 - \sin^2\theta_L}\right]^2 && (22)\\
&\rightarrow \begin{cases} E' = E \ \text{ for } \theta_L = 0 \\ E' = \left(\frac{A-1}{A+1}\right)^2 \equiv \alpha E \ \text{ for } \theta_L = \theta_{\max} \end{cases}
\end{aligned}
$$

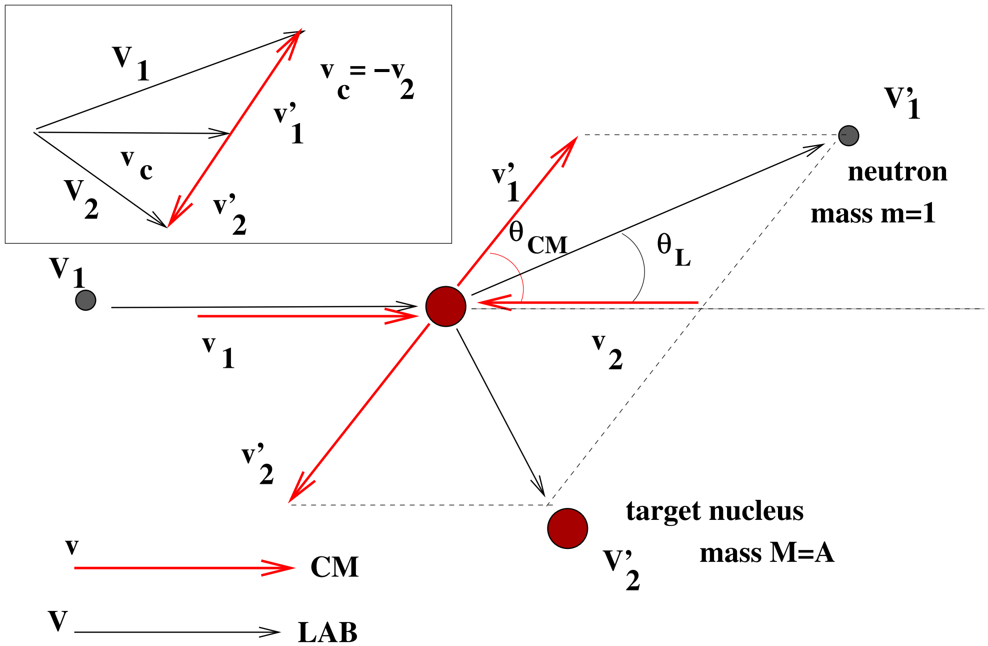
For $A > 1$ $\theta_{\max} = \pi$ (max energy loss)
For Hydrogen ($A = 1$) one has $\theta_{\max} = \pi/2$:

$$
E' = \frac{E}{4}\left[\cos\theta_L + \sqrt{1 - \sin^2\theta_L}\right]^2 = E\cos\theta_L \rightarrow \begin{cases} E' = E \ \text{for } \theta_L = 0 \\ E' = 0 \ \text{for } \theta_L = \frac{\pi}{2} \end{cases}
$$

Conclusion: the neutron max energy loss factor α
$(\Delta E = E - E' = (1-\alpha)E)$

$$
\alpha = \left(\frac{A-1}{A+1}\right)^2 \quad \text{is valid for any } A \qquad (23)
$$

Diffusion in the CM system



$$\boldsymbol{v}_c = \frac{\boldsymbol{V}_1 + A\boldsymbol{V}_2}{A+1} \;, \quad \boldsymbol{v}_c = -\boldsymbol{v}_2 \;, \quad \boldsymbol{v}_1 + A\boldsymbol{v}_2 = 0 \;\rightarrow$$

$$\rightarrow \boldsymbol{v}_1 = \frac{A}{A+1}\boldsymbol{V}_1 \qquad \boldsymbol{v}_2 = -\frac{1}{A+1}\boldsymbol{V}_1$$

$$\boldsymbol{V}_1' = \boldsymbol{v}_1' + \boldsymbol{v}_c = \boldsymbol{v}_1' - \boldsymbol{v}_2$$

Since $\cos(\pi - \theta) = -\cos\theta$**, from Carnot theorem**

$$V_1'^2 = \left[\frac{A}{A+1}V_1\right]^2 + \left[\frac{1}{A+1}V_1\right]^2 + 2V_1^2\frac{A}{(A+1)^2}\cos\theta_{\rm CM}$$

$$E' = E\frac{1}{(A+1)^2}(A^2 + 1 + 2A\cos\theta_{\rm CM}) \tag{24}$$

This is LAB energy as a function of $\theta_{\rm CM}$**. From (21), after some calculations:**

$$\cos\theta_L = \frac{A\cos\theta_{\rm CM} + 1}{\sqrt{A^2 + 1 + 2A\cos\theta_{\rm CM}}} \tag{25}$$

LAB and CM system

The CM system is useful because

- **from scattering theory we know the scattering angular distributions. For example, in S-wave the diffusion is isotropic.**
- **the elastic scattering distribution is monochromatic**

Hence, many quantities are calculated in CM and then transported in LAB, where we measure them.
In S-wave the scattering prob in CM is proportional to the solid angle (see also (24))

$$\frac{\mathrm{d}\omega}{4\pi} = \frac{-\,\mathrm{d}(\cos\theta_{\mathrm{CM}})}{2} = -\frac{(A+1)^2}{4A}\frac{\mathrm{d}E'}{E} \tag{26}$$

hence, all the $\cos\theta$ intervals are equally probable (isotropy)
After one collision the energy of the outcoming neutron in the LAB is equally probable within

$$\alpha E < E' < E \qquad \text{see} \quad (22) \text{ and } \quad (23)$$

For H, $0 < E' < E$. The mean energy after one collision is

$$\langle E' \rangle = \frac{\alpha E + E}{2} = \frac{1}{2}(1+\alpha)E$$

and the average and fractional average energy losses are:

$$\langle \Delta E \rangle = E - E' = \frac{1}{2}(1-\alpha)E \ , \quad \frac{\langle \Delta E \rangle}{E} = \frac{1}{2}(1-\alpha) \tag{27}$$

Slowing down and Lethargy

Nucleus	A	α	ξ
Hydrogen	1	0	1.000
Water	-	-	0.920
Deuterium	2	0.111	0.725
Beryllium	9	0.640	0.209
Carbon	12	0.716	0.158
Iron	56	0.931	0.0357
Uranium	238	0.983	0.00838

A useful quantity to describe slowing down is the lethargy

$$u = \ln(E_0/E) \tag{28}$$

where E is the current energy after the collision and E_0 is usually the highest (incoming or initial) energy.
During slowing down, the lethargy increases.
The average lethargy ξ in one collision is independent of the incoming energy (as in (27))
From (24) at page 70 and from (26):

$$\langle u \rangle \equiv \xi = \int_{\alpha E_0}^{E_0} \ln(E_0/E) \frac{\mathrm{d}\omega}{\mathrm{d}E} \mathrm{d}E \Big/ 4\pi \int_{\alpha E_0}^{E_0} \frac{\mathrm{d}\omega}{\mathrm{d}E} \mathrm{d}E \tag{29}$$

$$= \frac{(A+1)^2}{4AE_0} \int_{\alpha E_0}^{E_0} \ln(E_0/E)\, \mathrm{d}E = 1 + \frac{(A-1)^2}{2A} \ln \frac{A-1}{A+1}$$

Since

$$\begin{aligned}
\langle \ln(E_0/E_1) \rangle = \langle \ln E_0 - \ln E_1 \rangle &= \xi \\
\langle \ln(E_1/E_2) \rangle = \langle \ln E_1 - \ln E_2 \rangle &= \xi \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\langle \ln(E_{n-1}/E_n) \rangle = \langle \ln E_{n-1} - \ln E_n \rangle &= \xi
\end{aligned}$$

by summing up, after n collisions the mean energy is

$$\langle \ln E_n \rangle = \ln E_0 - n\,\xi \tag{30}$$

Slowing down and Lethargy: examples

A useful formula for $A > 1$ (compare with the previous table)

$$\xi = 1 + \frac{(A-1)^2}{2A} \ln \frac{A-1}{A+1} \simeq \frac{2}{A+2/3} \qquad (31)$$

- **an 1 MeV neutron is scattered through a 45^0 angle from a ^2H nucleus. Find**
  - **the energy of the scattered neutron:** **From (22)** $E' = 0.738$ **MeV**
  - **the recoil energy:** $E_A = E - E' = 1 - 0.738 = 0.262$ **MeV**
  - **the change in lethargy:**

$$\begin{aligned} \Delta u &= u' - u = \ln(E_0/E') - \ln(E_0/E) \\ &= \ln(E/E') = \ln(1/0.738) = 0.304 \end{aligned}$$

- **Calculate the mean number of collisions necessary to slow an 1 MeV neutron down to the thermal energy for H, ^2H, Water and C**

  From (30)

$$n = \frac{1}{\xi} \ln E/E_n$$

  Since $E = 1$ **MeV and** $E_n = 0.025$ **eV,**

$$n = \frac{1}{\xi} \ln(1\,000\,000/0.025) = \frac{1}{\xi} 17.5$$

  Using the ξ values for Hydrogen, Water, Deuterium and Carbon from the table at page 72 (or from (31) for D and C), we have

$$n(H) = 17.5 \ , \quad n(H_2O) = 19 \ , \quad n(D) = 24 \ , \quad n(C) = 110$$

0.38 Neutron diffusion

Let consider a neutron gas in a target medium.
The number of interactions per second n_a in a volume V is $F = \mathrm{d}n_a/\,\mathrm{d}V$ (interactions/cm^{3}s):

$$F = \frac{\mathrm{d}n_a}{\mathrm{d}V} = \int n(E)\, v(E)\, \Sigma(E)\, \mathrm{d}E = \int \Sigma(E)\, \Phi(E)\, \mathrm{d}E \quad (32)$$

where $n(E)$ is the neutron density, $v(E)$ their velocity and Φ (neutrons/s) is the total flux (called I in eq (2) at page 18)
In this case the one-dimensional classical diffusion law (Fick's law) holds

$$J = -D\frac{\mathrm{d}\Phi}{\mathrm{d}x} \quad (33)$$

where $\boldsymbol{J}$ is the neutron current density (neutron/ cm^2 s) and D (cm) is the diffusion coefficient.
Outside neutron physics Fick's law is often written as

$$J = -D\frac{\mathrm{d}n}{\mathrm{d}x} \quad (34)$$

where n =(number of particles/cm^3); in this case D (cm^2/s).
Fick's law is a universal law (physics, chemistry, biology, ..); it has a statistical origin, due to the difference in the (neutron) concentration.
There is no dynamical content in this law.
The 3-dimensional form is

$$\boldsymbol{J} = -D\boldsymbol{\nabla}\Phi$$

Continuity and diffusion equations

The continuity equation is simply the neutron balance in the medium:

Rate of change of the number of neutrons in V	=	production rate in V	-	absorption rate in V	-	rate of leakage in V

$$\frac{\partial n}{\partial t} = s - \Sigma_a \Phi - \boldsymbol{\nabla} \cdot \boldsymbol{J}$$

From Fick's law we obtain the diffusion equation

$$\frac{\partial n}{\partial t} = D\nabla^2\Phi - \Sigma_a \Phi + s \qquad (35)$$

In time independent problems (steady state situations) we can set $\frac{\partial n}{\partial t} = 0$

$$\nabla^2\Phi - \frac{\Phi}{L^2} = -\frac{s}{D}\ , \qquad L^2 = \frac{D}{\Sigma} \qquad (36)$$

where L is the diffusion length

Solutions of the diffusion equations I

For an infinite monochromatic planar source from (36) we have, at a distance x

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} - \frac{\Phi}{L^2} = 0$$

The general solution is

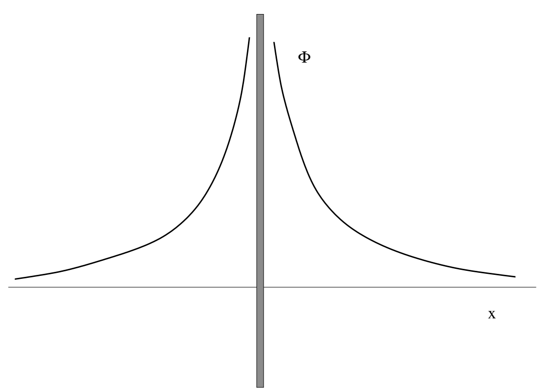
$$\Phi = A\,\mathrm{e}^{-x/L} + B\,\mathrm{e}^{x/L}$$

By discarding the solution increasing with x and using the boundary condition at the source plane

$$\lim_{x\to 0} J = S/2 \ , \ \text{ and } \quad J = -D\frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{DA}{L}\,\mathrm{e}^{-x/L}$$

one obtains

$$\Phi = \frac{SL}{2D}\,\mathrm{e}^{-|x|/L}$$



Solutions of the diffusion equations II

For a point source **we write eq (36) in spherical co-ordinates:**

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r} - \frac{1}{L^2}\Phi = 0$$

with general solution:

$$\Phi = A\frac{\mathrm{e}^{-r/L}}{r} + B\frac{\mathrm{e}^{r/L}}{r} \quad \text{and put } B = 0 \ .$$

The boundary source condition in this case is

$$\lim_{r\to 0} r^2 J(r) = \frac{s}{4\pi}$$

and from the equations

$$J = -D\frac{\mathrm{d}\Phi}{\mathrm{d}r} = DA\left(\frac{1}{rL} + \frac{1}{r^2}\right)\mathrm{e}^{-r/L} \ \rightarrow \ A = \frac{s}{4\pi D}$$

one obtains

$$\Phi = \frac{s}{4\pi D}\frac{\mathrm{e}^{-r/L}}{r} \ , \quad \Phi = n\,v \tag{37}$$

The neutrons absorbed per second n_a from (32) are

$$\mathrm{d}n_a = \Sigma_a \Phi(r)\,\mathrm{d}V = \Sigma_a \frac{s}{4\pi D}\frac{\mathrm{e}^{-r/L}}{r}4\pi r^2\,\mathrm{d}r = \frac{s}{L^2} r\,\mathrm{e}^{-r/L}\,\mathrm{d}r$$

By removing the source intensity s we obtain the absorption probability at r:

$$p(r)\,\mathrm{d}r = \frac{1}{L^2} r\,\mathrm{e}^{-r/L}\,\mathrm{d}r \tag{38}$$

The mean square absorption radius is

$$\langle r^2 \rangle = \int r^2 p(r)\,\mathrm{d}r = 6\,L^2 \tag{39}$$

Diffusion of thermal neutrons I

For thermal neutrons the diffusion equation

$$\nabla^2\Phi - \frac{\Phi}{L^2} = -\frac{s}{D}\ , \quad L^2 = \frac{D}{\Sigma} \tag{40}$$

should be integrated on the thermal spectrum $n(E)$ **from (1) of page 7 and on the velocity** $v(E) = \sqrt{2E/m}$**:**

$$\Phi_T = \int_0^\infty n(E)\, v(E)\, \mathrm{d}E = \frac{2n}{\sqrt{\pi}}\sqrt{\frac{2kT}{m}}$$

where n **is the total number of thermal neutrons. Remember the thermal equivalences (see page 8)**

$$E_T = kT = \frac{1}{2}mv_T^2 = 8.617\ T\, 10^{-5}\ \mathrm{eV} = \frac{T}{11.6}\,\mathrm{meV}$$

and the standard flux parameters at 2200 m/s:

$$v_0 = 2\,200\ \mathrm{m/s}\ ,\ \ E_0 = 25.3\ \mathrm{meV}\ ,\ \ T_0 = 293.61\ {}^0\mathrm{C}$$

It is easy to show that:

$$\frac{\Phi_0}{\Phi_T} = \frac{\sqrt{\pi}}{2}\frac{v_0}{v_T} = \frac{\sqrt{\pi}}{2}\sqrt{\frac{T_0}{T}}$$

By defining the average quantities over the thermal spectrum and using (19) of page 64 in the case of $1/v$ **absorption:**

$$\begin{aligned} \langle\Sigma_a\rangle &= \frac{1}{\Phi_T}\int \Sigma_a(E)\Phi(E)\,\mathrm{d}E = g(T)\,\Sigma_a(E_0)\,\Phi_0/\Phi_T \\ \langle D\rangle &= \frac{1}{\Phi_T}\int D(E)\Phi(E)\,\mathrm{d}E \end{aligned}$$

the thermal diffusion equation becomes

$$\langle D\rangle\,\nabla^2\Phi - \langle\Sigma\rangle\,\Phi_T = -s_T\ , \quad L_T^2 = \frac{\langle D\rangle}{\langle\Sigma\rangle} \tag{41}$$

where s_T **is the total flux of thermal neutrons.**

Diffusion of thermal neutrons II

Exercise

Moderator	Density g/cm^3	$\langle D \rangle$ cm	$\langle \Sigma \rangle$ cm^{-1}	L_T^2 cm^2	L_T cm
H_2O	1.00	0.16	0.0197	8.1	2.85
D_2O	1.10	0.87	2.9×10^{-5}	2.9×10^4	170
Be	1.85	0.50	1.04×10^{-3}	480	21
Graphite	1.60	0.84	2.4×10^{-4}	3500	59

Table 1: Thermal neutron diffusion parameters for some moderators at 20 ^{0}C. Note the low absorption by the heavy water.

A point source emits 10^7 thermal neutrons/s in water at room temperature. Calculate the flux at 15 cm from the source and the root mean square absorption radius. Calculate the same quantities also for the heavy water.

Solution

The flux from a point source is given by (37), with the values averaged over the thermal spectrum:

$$\Phi = \frac{s}{4\pi \langle D \rangle} \frac{e^{-r/L_T}}{r} .$$

From the table above, $L_T = 2.85$ cm and $\langle D \rangle = 0.16$ cm. With $s = 10^7$ and $r = 15$ cm we obtain:

$$\Phi_T = \frac{10^7 \; e^{-15/2.85}}{4\pi \times 0.16 \times 15} = 1.72 \times 10^3 \quad \text{neutrons/cm}^2\text{s}$$

The root mean square absorption radius is from (39):

$$\sqrt{\langle r^2 \rangle} = \sqrt{6 L_t^2} = \sqrt{6 \times 8.1} = 7.0 \text{ cm.}$$

The same calculation for the heavy water gives

$$\Phi = 5.57 \times 10^4 \quad \text{neutrons/cm}^2\text{s} \; , \quad \sqrt{\langle r^2 \rangle} = 416 \text{ cm}$$

0.39 Piles and reactors

Some important parameters

- τ: reactor cycle or generation time

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{n(k-1)}{\tau} \ , \quad n(t) = n_0 \mathrm{e}^{(k-1)/\tau}$$

- σ_r: neutron absorption cross section (fission excluded)
- σ_f: neutron fission cross section
- $\sigma_a = \sigma_r + \sigma_f$: total absorption cross section
- σ_r / σ_f: ratio α
- ν: average neutrons per fission ($\simeq 2.5$ in U)
- η: average neutrons per absorption $\nu \, \sigma_f / \sigma_a$
- f: neutron fraction absorbed by the fuel
- k: neutron chain moltiplication factor
- k_{eff}: effective moltiplication factor (see next)
- ϵ: (neutrons per fission)/(neutrons per thermal fission) ($\simeq 1.04$)
- p: probability of a neutron capture followed by fission

The four factor formula holds:

$$k = \eta \, f \, \epsilon \, p \tag{42}$$

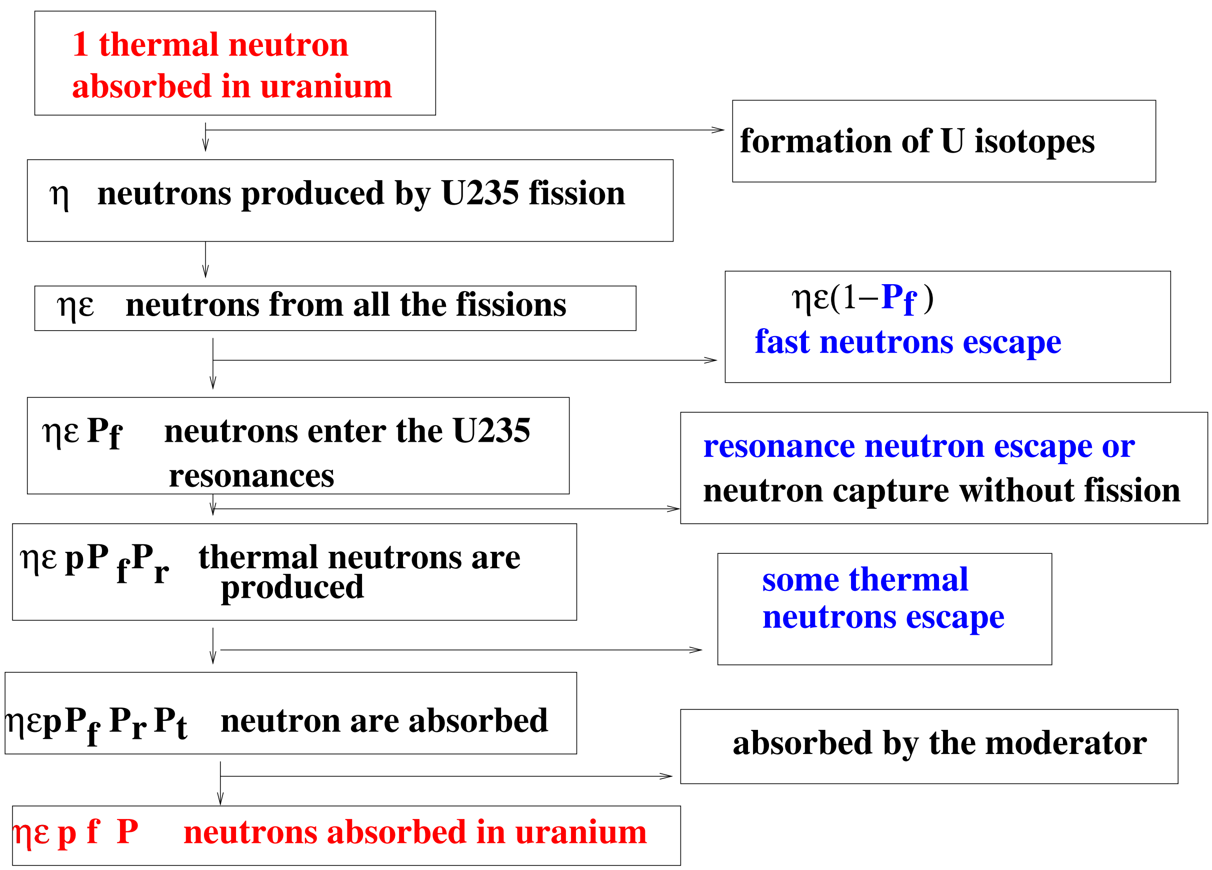
Nuclear reactor balance

The effective four factor formula has also a factor $P = P_f P_r P_t$ including the fast neutron $(1 - P_f)$, resonance neutron $(1 - P_r)$ and thermal neutrons $(1 - P_t)$ escapes from the reactor

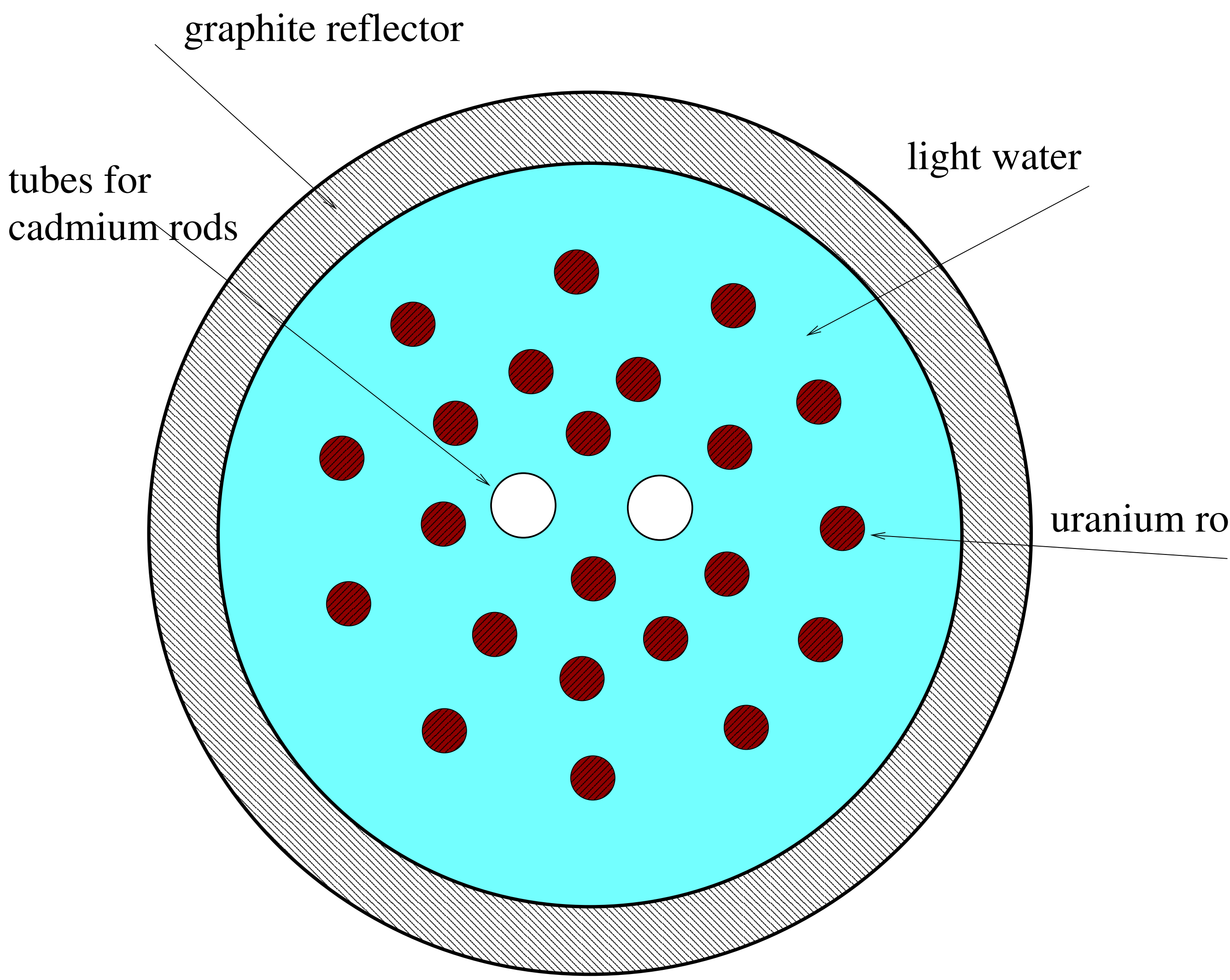
$$k_{\text{eff}} = \eta\, \epsilon\, p\, f\, P$$

$k_{\text{eff}} > 1$ for a critical assembly (reactor)
Some values for a critical pile:

$$\epsilon = 1.038 \quad p = 0.905\ , \quad f = 0.888 \quad \eta = 1.308 \rightarrow k = \eta\, \epsilon\, p\, f = 1.081$$



Triga Mark Reactor



0.40 The Monte Carlo method

- **It originates at Los Alamos by an idea of J. von Neumann and S.Ulam, to treat scattering and absorption of neutron in fissile materials.**
- **from the 50-ies there is a big diffusion of the method, thanks to computers**
- **presently it is one of the more important methods of nuclear physics**
- **the method is based on two fundamental ideas: the cumulative variable theorem, and the Central Limit theorem. Based on this, only an uniform random number generator is necessary**

$$\xi \sim U(0, 1)$$

0.41 The sampling technique

Theorem 1:
If $a \le X \le b$**, e** $X \sim U(a,b)$

$$P\{x_1 \le X \le x_2\} = \frac{1}{b-a}\int_{x_1}^{x_2} \mathrm{d}x = \frac{x_2 - x_1}{b-a} \ . \qquad (1)$$

If (1) is valid, then $X \sim U(a,b)$
Theorem 2:
If X **has a continuous density p(x) the cumulative random variable**

$$C(X) = \int_{-\infty}^{X} p(x)\,\mathrm{d}x$$

is uniform in $[0,1]$**, that is** $C \sim U(0,1)$**.**

Example: simulate a nuclear event when $\sigma_{\mathrm{sc}} = 2$ **barn and** $\sigma_{\mathrm{abs}} = 3$ **barn.**
If `RANDOM` < 0.4 **there is scattering, otherwise absorption occurs.**

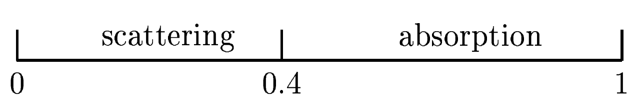


Figure 1: Simulation of the scattering-absorption mechanism with the routine `rndm`.

The sampling technique

Exercise: generate nuclear events at the point

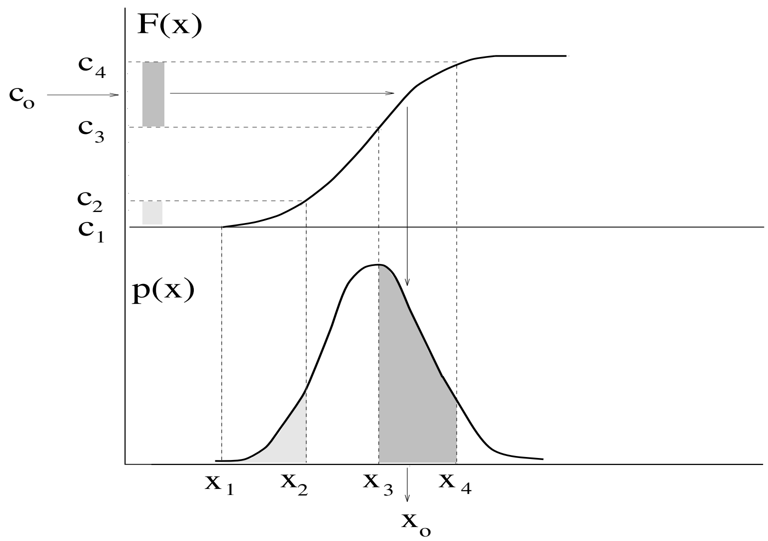


Figure 2: The cumulative variable theorem.

x by knowing the cross section σ, or $\Sigma = \sigma N$.
From page 19:

$$F(x) = \int_0^x \Sigma\ \mathrm{e}^{-\Sigma x} = 1 - e^{-\Sigma x} = \texttt{RANDOM}$$

by inversion:

$$x = -\frac{1}{\Sigma} \ln(1 - \texttt{RANDOM})$$

or:

$$x = -\frac{1}{\Sigma} \ln(\texttt{RANDOM}) \tag{43}$$

General rule: the number of mean free paths Σx is a random variable $-\ln(\texttt{RANDOM})$

0.42 Rejection method

$$\begin{cases} x_i &= a + \xi_1(b-a) \\ y_i &= \xi_2 h \end{cases}$$

with $0 \le \xi_1, \xi_2 \le 1$.

Accept x_i **if** $y_i < p(x_i)$

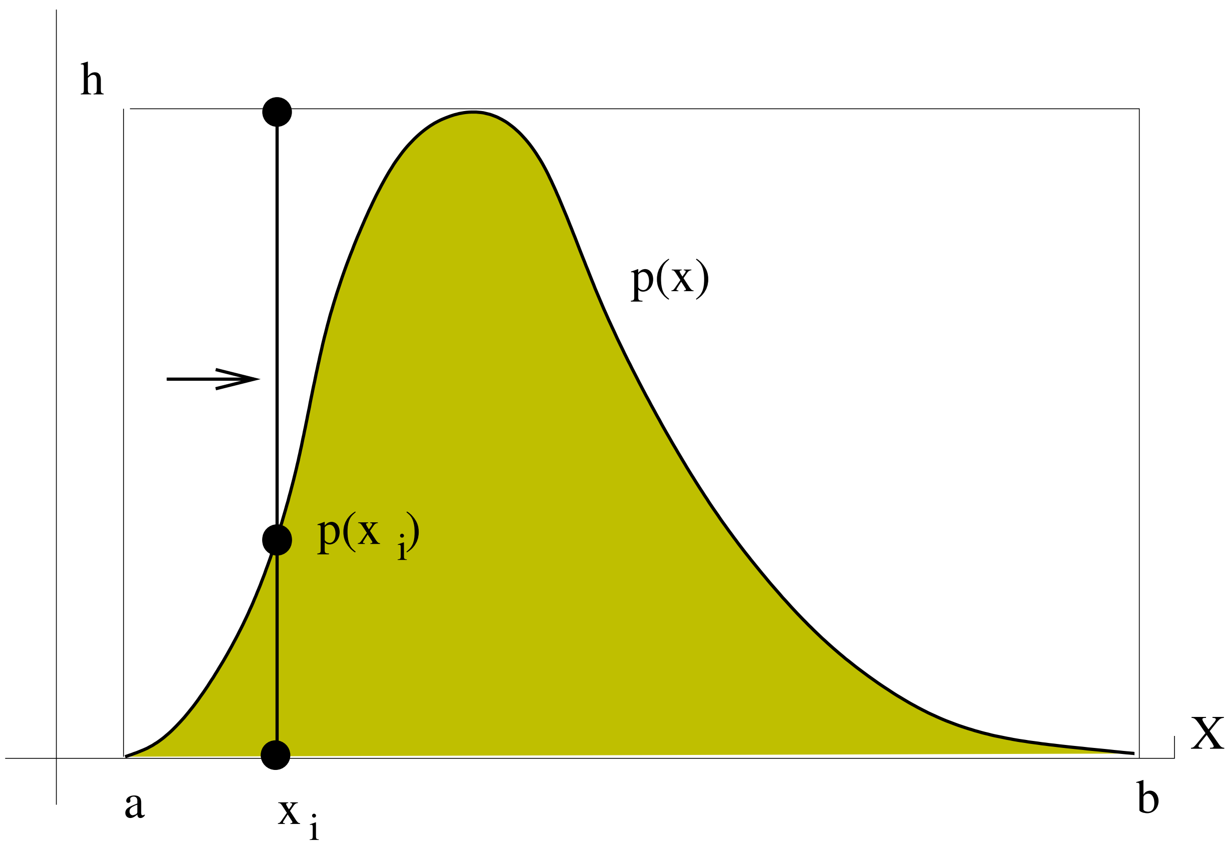


Figure 3: The rejection technique.

0.43 Sampling examples

Uniform sampling into a circle:

$$p(\varphi)\,\mathrm{d}\varphi = \frac{\rho R^2\,\mathrm{d}\varphi/2}{\rho\pi R^2} = \frac{\mathrm{d}\varphi}{2\pi} \ .$$

For $q(r)$ we have

$$q(r)\,\mathrm{d}r = \frac{\rho 2\pi r\,\mathrm{d}r}{\rho\pi R^2} = \frac{2r}{R^2}\,\mathrm{d}r \ .$$

The corresponding cumulatives are:

$$\xi_1 = P(\varphi) = \int_0^{\varphi} p(\varphi)\,\mathrm{d}\varphi = \frac{\varphi}{2\pi} \ ,$$

$$\xi_2 = Q(r) = \int_0^{r} q(r)\,\mathrm{d}r = \frac{r^2}{R^2} \ .$$

$$\begin{cases} \varphi &= 2\pi\xi_1 \\ r &= R\sqrt{\xi_2} \ . \end{cases}$$

Sampling examples

Isotropy

$$\begin{cases} x &= R \,\mathrm{sen}\,\vartheta \cos\varphi \\ y &= R \,\mathrm{sen}\,\vartheta \,\mathrm{sen}\,\varphi \\ z &= R \cos\vartheta \ . \end{cases}$$

$$\mathrm{d}\Omega = \mathrm{sen}\,\vartheta \,\mathrm{d}\vartheta \,\mathrm{d}\varphi \ .$$

If $n_{\rm tot}$ is the number of points:

$$\frac{n_{tot}}{4\pi} = \frac{\mathrm{d}n}{\mathrm{d}\Omega} \ , \qquad \text{isotropy}$$

We have:

$$p(\Omega)\,\mathrm{d}\Omega = \frac{\mathrm{d}n}{n_{tot}} = \frac{\mathrm{d}\Omega}{4\pi} = \frac{\mathrm{sen}\,\vartheta \,\mathrm{d}\vartheta \,\mathrm{d}\varphi}{4\pi}$$

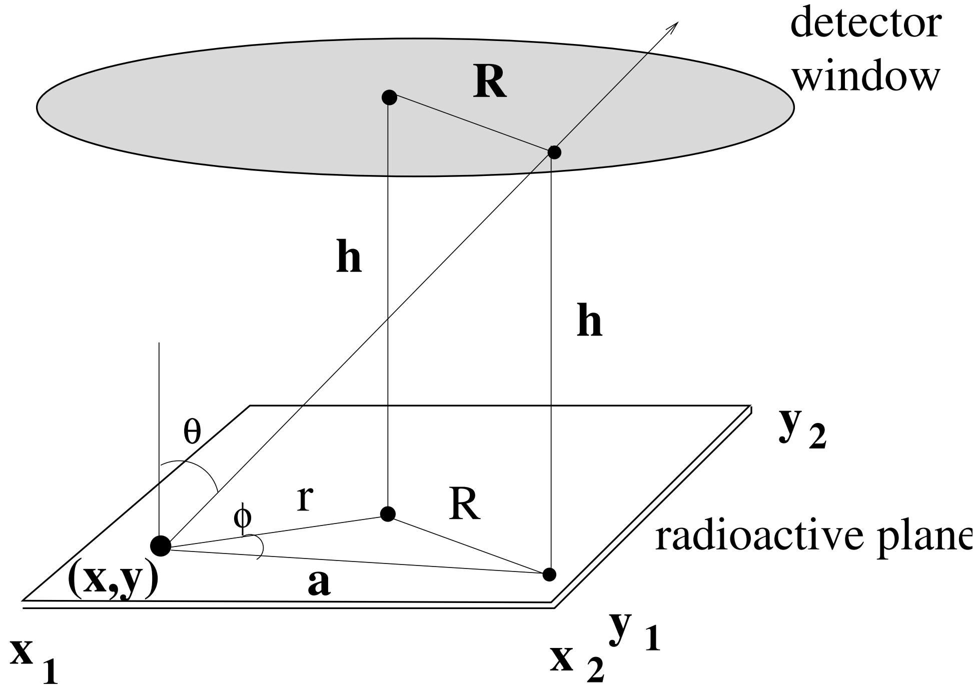
$$\begin{aligned} p(\varphi)\,\mathrm{d}\varphi &= \frac{1}{4\pi}\mathrm{d}\varphi \int_0^{\pi} \mathrm{sen}\,\vartheta \,\mathrm{d}\vartheta = \frac{1}{2\pi}\mathrm{d}\varphi \ , \\ q(\vartheta)\,\mathrm{d}\vartheta &= \frac{1}{4\pi}\mathrm{sen}\,\vartheta \,\mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi = \frac{\mathrm{sen}\,\vartheta}{2}\mathrm{d}\vartheta \ . \end{aligned}$$

The corresponding cumulatives are:

$$\begin{aligned} \xi_1 = P(\varphi) &= \frac{\varphi}{2\pi} \ , \\ \xi_2 = Q(\vartheta) &= \frac{1-\cos\vartheta}{2} \end{aligned}$$

$$\begin{cases} \varphi &= 2\pi\xi_1 \\ \vartheta &= \mathrm{acos}(1-2\xi_2) \ . \end{cases}$$

Detector efficiency



ξ **is** $0 < RANDOM < 1$

$$x = x_1 + \xi(x_2 - x_1) \ , \quad y = y_1 + \xi(y_2 - y_1)$$

$$\cos\theta = 1 - 2\xi \ , \quad \phi = 2\pi\,\xi$$

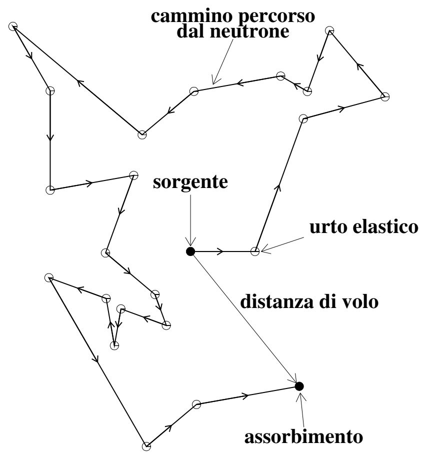
$$a = h\,\mathrm{tg}\theta \ , \quad r = \sqrt{x^2 + y^2} \ , \quad R = \sqrt{r^2 + a^2 - 2ra\cos\phi}$$

if R **is less then the detector radius** R_d **the particle is counted. Efficiency:**

$$\epsilon = \frac{\text{particles with } \mathrm{R} < \mathrm{R_d}}{\text{generated particles}}$$

0.44 Neutron diffusion: MC method

Neutron diffusion from a point source into a ^{12}C sphere



- **Interaction: constant cross sections**

$$\Sigma_T = \Sigma_a + \Sigma_{el}$$

- **Kinematics: the source emits isotropically in the LAB:**

$$\phi = 2\pi\,\xi_1 \;, \qquad \cos\theta = 1 - 2\,\xi_2$$

c.m. director cosines:

$$\begin{aligned}\alpha &= \sin\theta\,\cos\phi \\ \beta &= \sin\theta\,\cos\phi \\ \gamma &= \cos\theta\end{aligned}$$

Neutron diffusion: MC method II

- **Sampling of the interaction point:**
  probability density:

$$p(x)\,\mathrm{d}x = \Sigma_T\ \exp(-x\Sigma_T)\,\mathrm{d}x$$

from the general rule (43) of page 85 distance between two successive interactions:

$$d = -\frac{1}{\Sigma_T}\ \ln \xi_3 \qquad (44)$$

- **Type of interaction:**

$$0 \le \xi_4 < \Sigma_a/\Sigma_T \qquad \text{absorption} \ ,$$
$$\Sigma_a/\Sigma_T \le \xi_4 \le 1 \qquad \text{elastic scattering} \ ;$$

- **New flight direction (if elastic scattering):**
  In S wave the neutron scattering is isotropic in the c.m. frame:

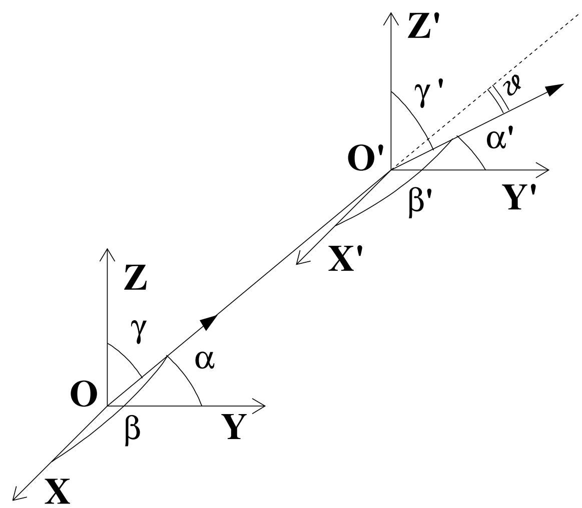
$$\cos\theta_{cm} = 1 - 2\ \cos\xi_5\ , \qquad \phi_{cm} = 2\pi\,\xi_6$$

Some formulae (see (25) at page 70) are used to transform into the LAB system:

$$\cos\theta = \frac{1 + A\cos\theta_{cm}}{\sqrt{A^2 + 2A\cos\theta_{cm} + 1}}$$
$$\phi = \phi_{cm}$$

where A (= 12 in this case) is the atomic weight.

Neutron diffusion: MC method III



The new director cosines α', β', γ' can be found with the formula

(see http://www.springer.it/libri_libro.asp?id=314):

$$\begin{aligned} \alpha' &= \mu\alpha + a(\alpha\gamma\sin\phi + \beta\cos\phi) \\ \beta' &= \mu\beta + a(\beta\gamma\sin\phi - \alpha\cos\phi) \\ \gamma' &= \mu\gamma - a(1-\gamma^2)\sin\phi \end{aligned}$$

where

$$a = \sqrt{\frac{1-\mu^2}{1-\gamma^2}} \ , \quad \mu = \cos\theta \ , \quad |\gamma| \neq 1 \ .$$

With $|\gamma| = 1$:

$$\alpha' = \gamma\sqrt{1-\mu^2}\cos\phi \ , \quad \beta' = \sqrt{1-\mu^2}\sin\phi \ , \quad \gamma' = \gamma\,\mu$$

Neutron diffusion: MC method IV

With the distance d and the current director cosines we can update the neutron coordinates

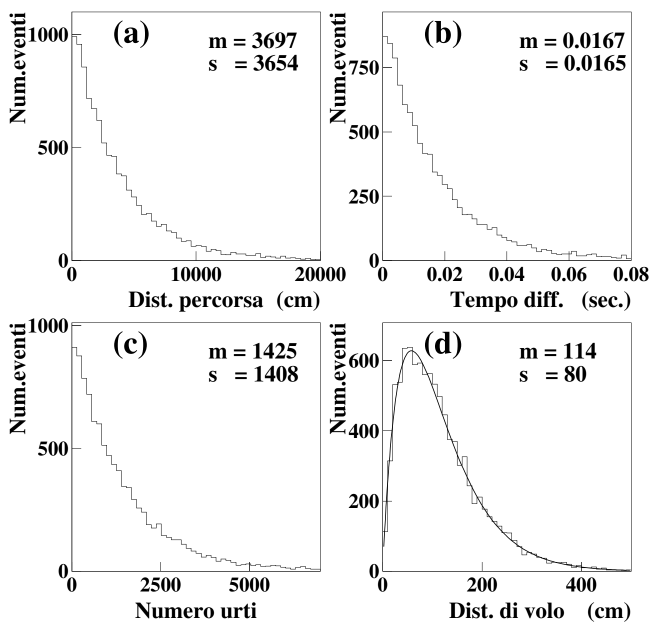
- **Distance**: $d_i = d + d_{i-1}$ where d is from (44)
- **Coordinates**: at the end these gives path and distance

$$x_i = x_{i-1} + d_i\alpha \ , \quad y_i = y_{i-1} + d_i\beta \ , \quad z_i = z_{i-1} + d_i\gamma$$

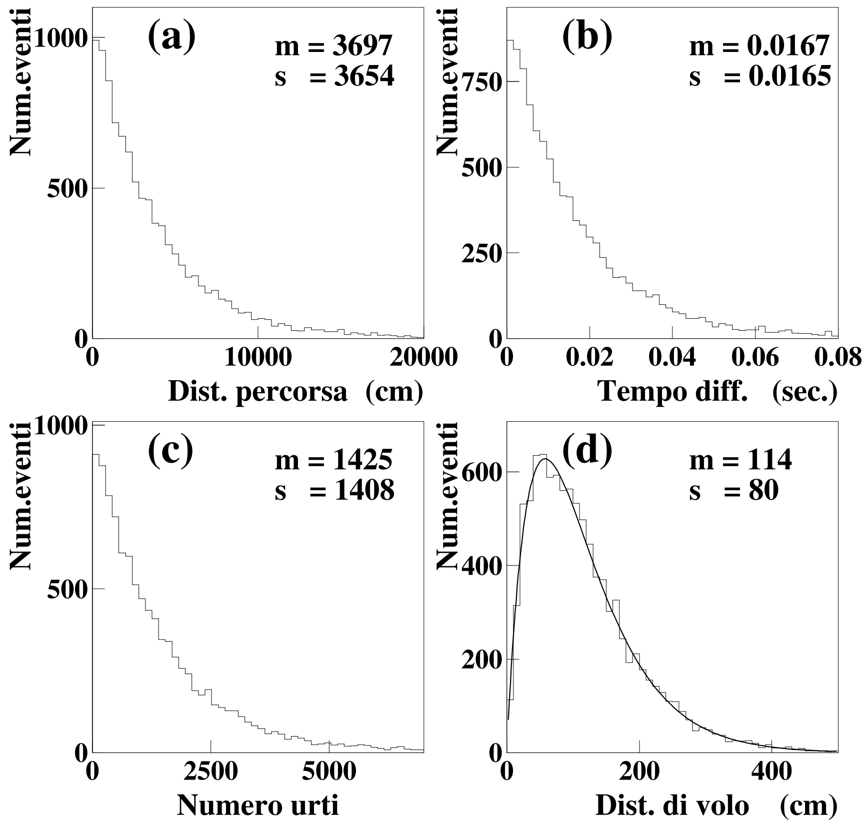
- **Time of flight**: (at $E = 0.025$ eV $v = 2200$ m/s);

$$t_i = t_{i-1} + d_i/v$$

- **Number of collisions**: simply by counting the interaction points until the neutron is absorbed.



Neutron diffusion: MC method V



Interpretation of the MC results

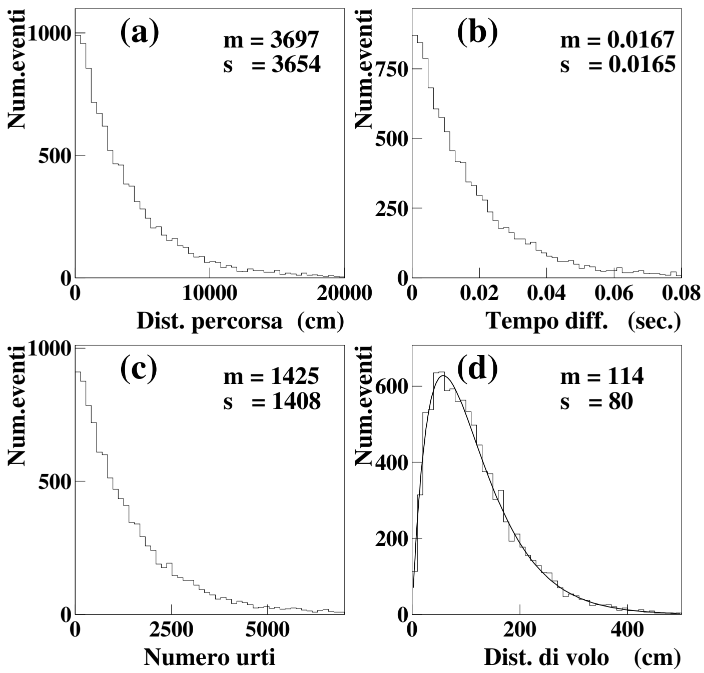
- Total zig-zag distance (a)
  this is the exponential distribution

$$p(x) = \Sigma_a \, \mathrm{e}^{-x\Sigma_a}$$

- Diffusion time (b)
  Same as in (a) with a change of variable, because v is kept constant:

$$p(t) = v \, \Sigma_a \, \mathrm{e}^{-tv\Sigma_a}$$

Neutron diffusion: MC method V



- **Number of collisions (c)**
  This distribution is not poissonian, because the events are correlated, but geometrical, which is exponential for a high number of collisions.

- **Source-absorption straight distance (d)**
  This is a solution of the diffusion equation (38) of page 77 (see the fit in fig.(d)):

$$p(r)\,\mathrm{d}r = \frac{r}{L^2}\,\mathrm{e}^{-r/L}\,\mathrm{d}r$$

Neutron Scilab code

```
// ---------------------------------------------------------------
//
// codice SCILAB Neutroni: calcolo delle distribuzioni
// caratteristiche della diffusione di neutroni in un mezzo
//
// Le quantita' di input sono descritte interattivamente
// Se la sezione d'urto di assorbimento e' piccola come nel caso del
// Carbonio, il programma puo' durare molti secondi per evento
//
// Autore: A. Rotondi 13 aprile 2004
//
// ----------------------------------------------------------------

// routine richieste
exec(macroscil+'Histplote.sci');
exec(macroscil+'Histfreqe.sci');


pig= 3.14159265;

// Calcolo dei coseni direttori delle direzioni di volo

function   x=Coseni(mass, cd);
  fi = 2.*pig*grand(1,1,'def');
  sfi = sin(fi);
  cfi = cos(fi);
  coscm = 1. - 2.*grand(1,1,'def');
  coslab = (1.+mass*coscm)/sqrt(mass*mass+2.*mass*coscm+1.);
  mu = sqrt(1.-coslab*coslab);
  if(abs(cd(3))==1.) then
    x(1)=mu*cfi; x(2)=mu*sfi; x(3)=cd(3)*coslab;
  else
    ck = mu/sqrt(1.-cd(3)*cd(3));
    x(1) = coslab*cd(1) + ck*(cd(1)*cd(3)*sfi + cd(2)*cfi);
    x(2) = coslab*cd(2) + ck*(cd(2)*cd(3)*sfi - cd(1)*cfi);
    x(3) = coslab*cd(3) - ck*(1.-cd(3)*cd(3))*sfi;
```

```
   end;
 endfunction;

// Codice principale

// quantita' di input
   amass = input("massa del nucleo (unita atomiche)..
   (esempio: digitare  12 per il Carbonio).");
   sigel = input("sigma macroscopica elastica (1/cm) (es.: 0.3851");
   sigas = input("sigma macroscopica assorbimento (1/cm) (es.: 0.0002728)");
   vel   = input("velocita (m/s) (es.: 2200)");
   vel   = vel*100;  // velocita' in cm al secondo
   Nevt  = input("numero eventi (es.: 10000");

   sigtot = sigel + sigas;
   sigper = sigel/sigtot;

// azzeramento vettori da istogrammare

   Disper = zeros(1,Nevt);
   Timvol = zeros(1,Nevt);
   Nurti  = zeros(1,Nevt);
   Disvol = zeros(1,Nevt);
   x      = zeros(1,3);
   pstep = zeros(1,3);
   cd     = zeros(1,3);

// ciclo di generazione dei neutroni

   for k=1:Nevt,
      Disper(k)=0;
      Nurti(k)=0;
      for i=1:3, pstep(i)=0; end;
      fi = 2.*pig*grand(1,1,'def');
      cd(3) = 1.-2.*grand(1,1,'def');
      cd(1) = sqrt(1.-cd(3)*cd(3))*cos(fi);
      cd(2) = sqrt(1.-cd(3)*cd(3))*sin(fi);
      iflg=0;
```

```
    while (iflg==0),
        camm = -log(grand(1,1,'def'))/sigtot;
        for j=1:3, pstep(j)=pstep(j)+camm*cd(j); end;
        Disper(k)=Disper(k)+camm;
        Timvol(k)=1000.*Disper(k)/vel; // millisecondi
        Nurti(k)=Nurti(k)+1;
        if(grand(1,1,'def') > sigper) then,
             Disvol(k)=sqrt(pstep(1)^2+pstep(2)^2+pstep(3)^2);
             iflg=1;
        else
             x=Coseni(amass,cd);
             cd(1)=x(1); cd(2)=x(2); cd(3)=x(3);
        end;
    end;
    // stampa intermedia del numero di eventi
    if(modulo(k,10) ==0) then,
     write(%io(2)," numero di neutroni: "+string(k));
    end;
 end;

// display dei risultati alla fine

 xbasc();
 subplot(2,2,1)
 Histplote(30,Disper,errors=1,stat=1); xtitle('distanza zig-zag (cm)');
 subplot(2,2,2)
 Histplote(30,Timvol,errors=1,stat=1); xtitle('tempo (ms)');
 subplot(2,2,3)
 sp= [min(Nurti):max(Nurti)];
 [ind Occ]=dsearch(Nurti,sp,'d');  // Occ=frequenze per spettro DISCRETO
 Histfreqe(Occ,sp,errors=1,stat=1); xtitle('numero urti');
 subplot(2,2,4)
 Histplote(50,Disvol,errors=1,stat=1); xtitle('distanza di volo (cm)');
```

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