A fast introduction to the tracking and to the Kalman filter

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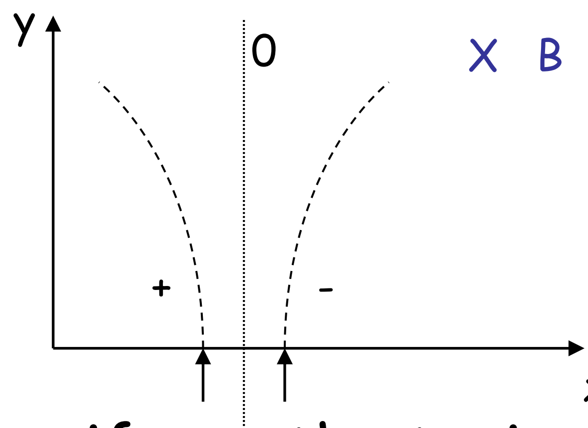
The tracking

To reconstruct the particle path to find the origin (vertex) and the momentum

The trajectory is usually curved by the Lorentz force



$$\underline{F} = \frac{q}{c}\left(\underline{v} \times \underline{B}\right)$$

Even when B is uniform, the trajectory is NOT an helix, due to

- energy loss
- multiple scattering

The track is defined as a set of points usually on detector planes (real and/or virtual)

The track

the five track coordinates: 1/p, v',w',v,w

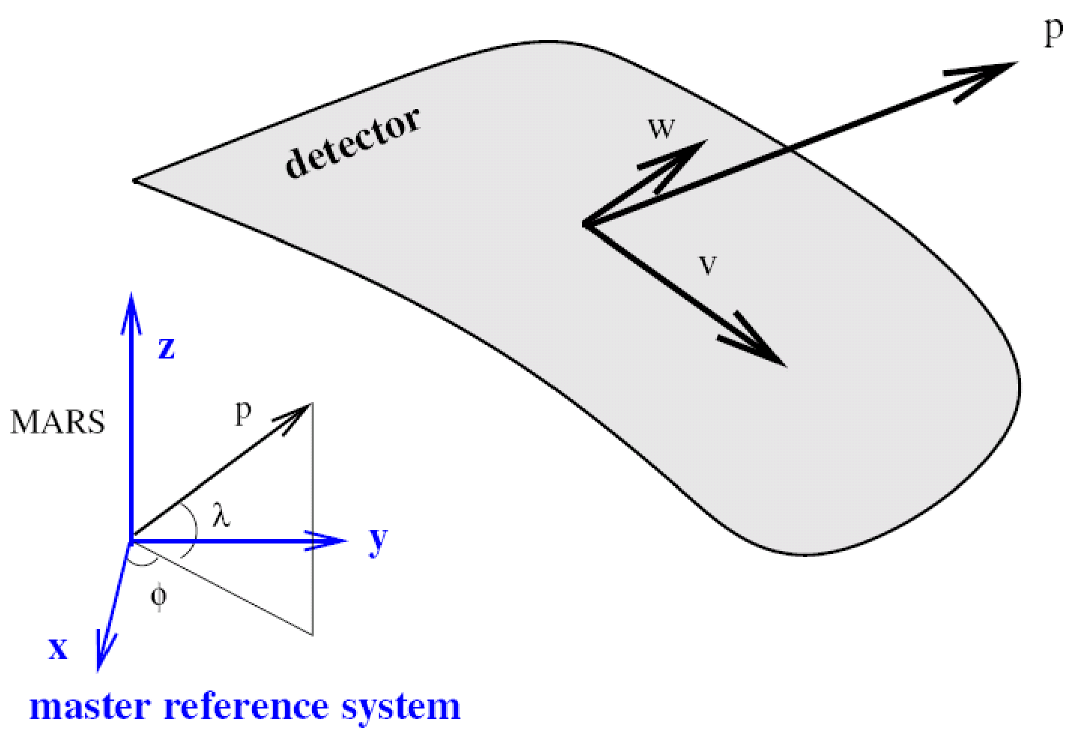


Figure 1: The five track parameters

Since on a detector plane we have two coordinates (v,w) and three momentum components (p_u, p_v, p_w) the track

is a 5-dimensional mathematical entity

Fitting method	Helix	Spline	Kalman
Magnetic field dishomogeneity Material effect	NO NO	YES NO	YES YES

Tracking neglecting inhomogeneous magnetic field and the medium effects

Global fit HELIX

Tracking in inhomogeneous magnetic field neglecting the medium effects

Global fit SPLINES

H. Wind, NIM 115(1974)431

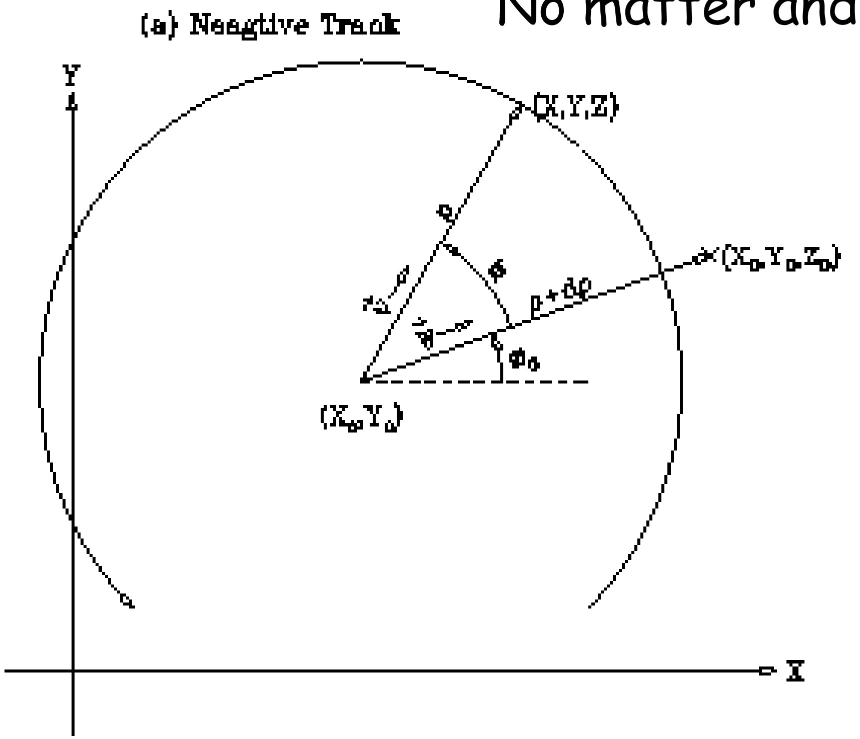
Tracking in inhomogeneous magnetic field with energy loss and multiple scattering

Progressive fit KALMAN ...

R. Frühwirth, NIM A262(1987)444

The Helix

No matter and uniform magnetic field



$$x(s) = x_0 + R_H\left[\cos\left(\Phi_0 + h\,s\,\frac{\cos\lambda}{R_H}\right) - \cos\Phi_0\right]$$

$$y(s) = y_0 + R_H\left[\sin\left(\Phi_0 + h\,s\,\frac{\cos\lambda}{R_H}\right) - \sin\Phi_0\right]$$

$$z(s) = z_0 + s\sin\lambda$$

s = track length
P(x_0, y_0, z_0) = starting point
λ = dip angle
Φ_0 = azimuthal angle
R_H = radius
h = -*sign*(qB_z)

B // z → two planes:
xy : circle
z - s : straight line

$$s = \frac{R}{h\cos\lambda}\left[\arctan\left(\frac{R\sin\Phi_0 + (y - y_0)}{R\cos\Phi_0 + (x - x_0)}\right) - \Phi_0\right]$$

Spline fit

H. Wind, NIM 115 (1974), 431

No medium effects, dishomogeneous magnetic field is taken intoaccount

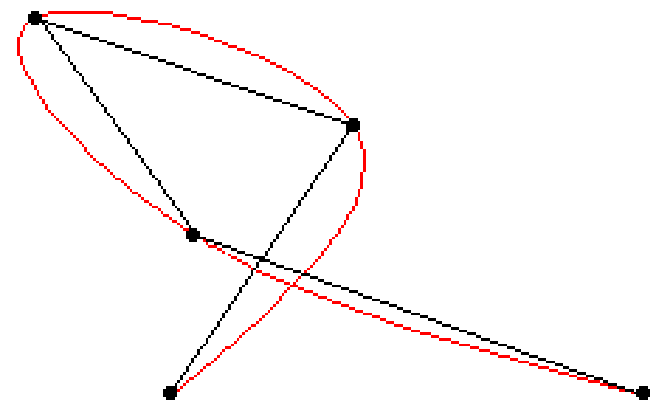
- The spline is a smooth segmented polynomial

- Cubic spline through *n+1* points $y_0, \ldots, y_n$:

$$Y_i(t) = a_i + b_i t + c_i t^2 + d_i t^3$$

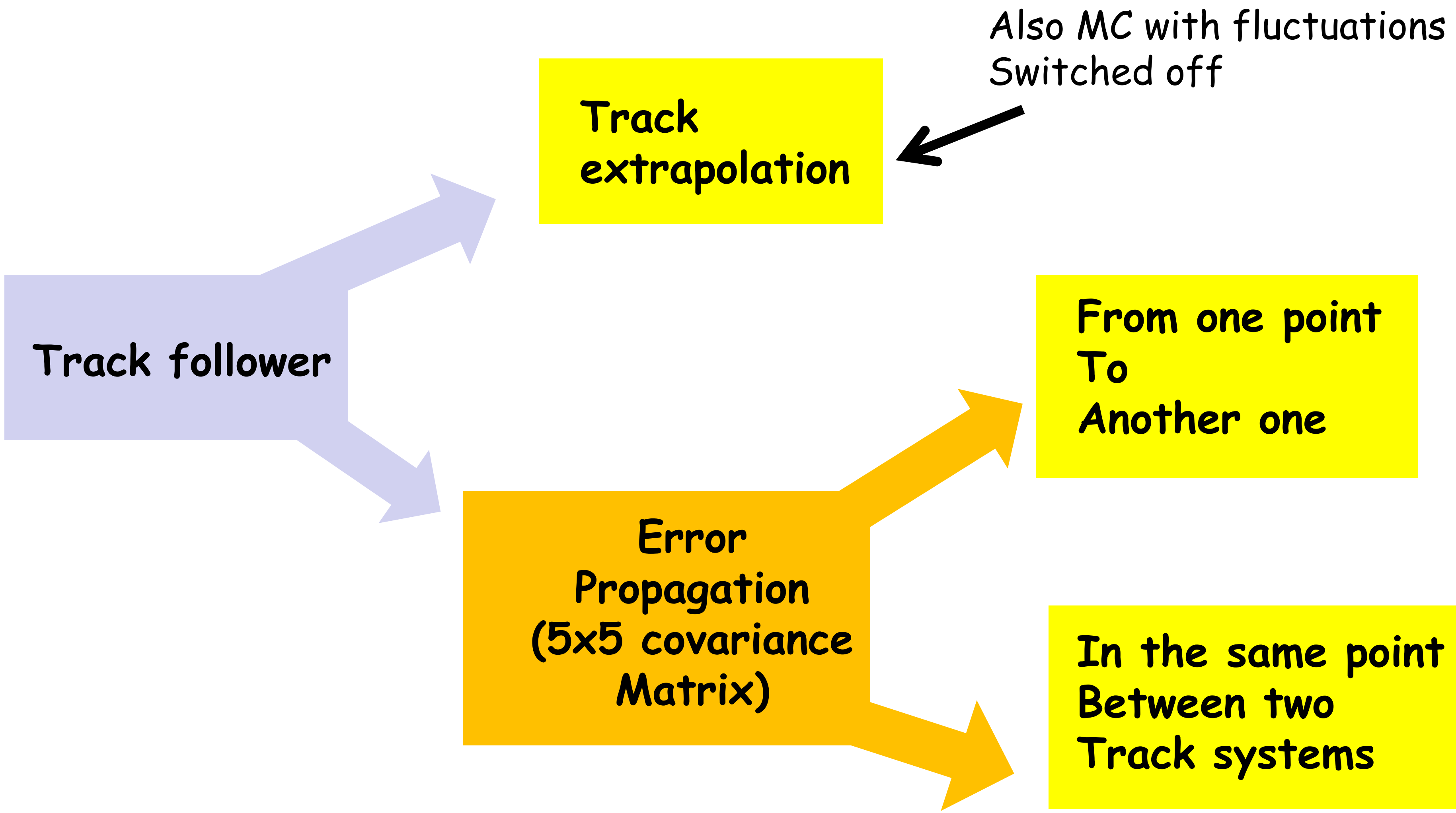
parameters

$i = 0,\ldots,n$
$t \in [0,1]$



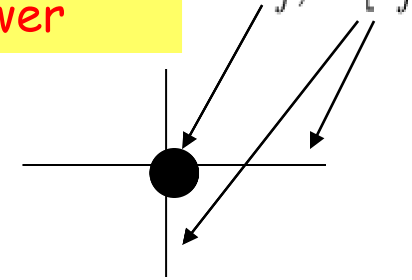
- The parameters are found by constraining the pieces of splines to be connected in the measured points assuring the continuity up to the 2 derivative

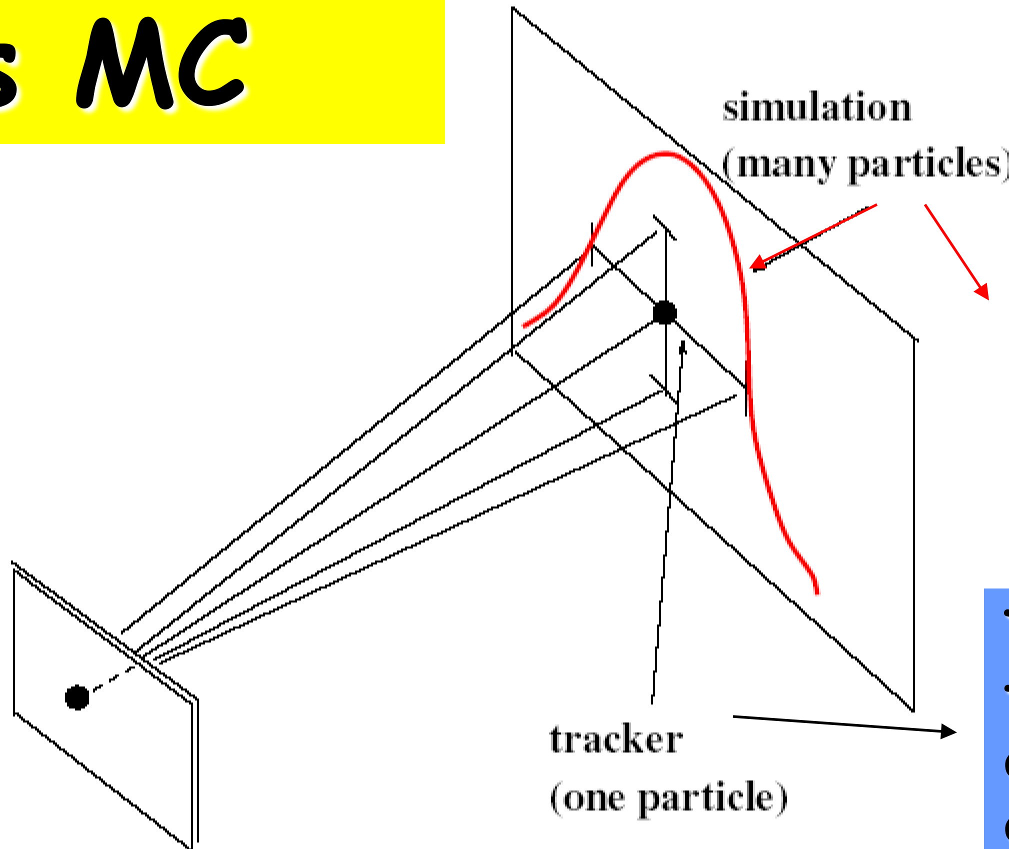
What is the track follower?



Tracking vs MC

$e_i, \boldsymbol{\sigma}[e_i]\,\Delta x$, medium $\rightarrow$ **track follower** $\rightarrow e_j, \boldsymbol{\sigma}[e_j]$





MC= at each step the trajectory is sampled as a random value

Tracking= at each step the trajectory is calculated as a mean value with an associate error

Energy loss affects both tracking (averages) and error propagation (covariance matrix), multiple scattering affects the error propagation only.

GEANE

V. Innocente et al. *Average Tracking and Error Propagation Package*, CERN Program Library W5013-E (1991).

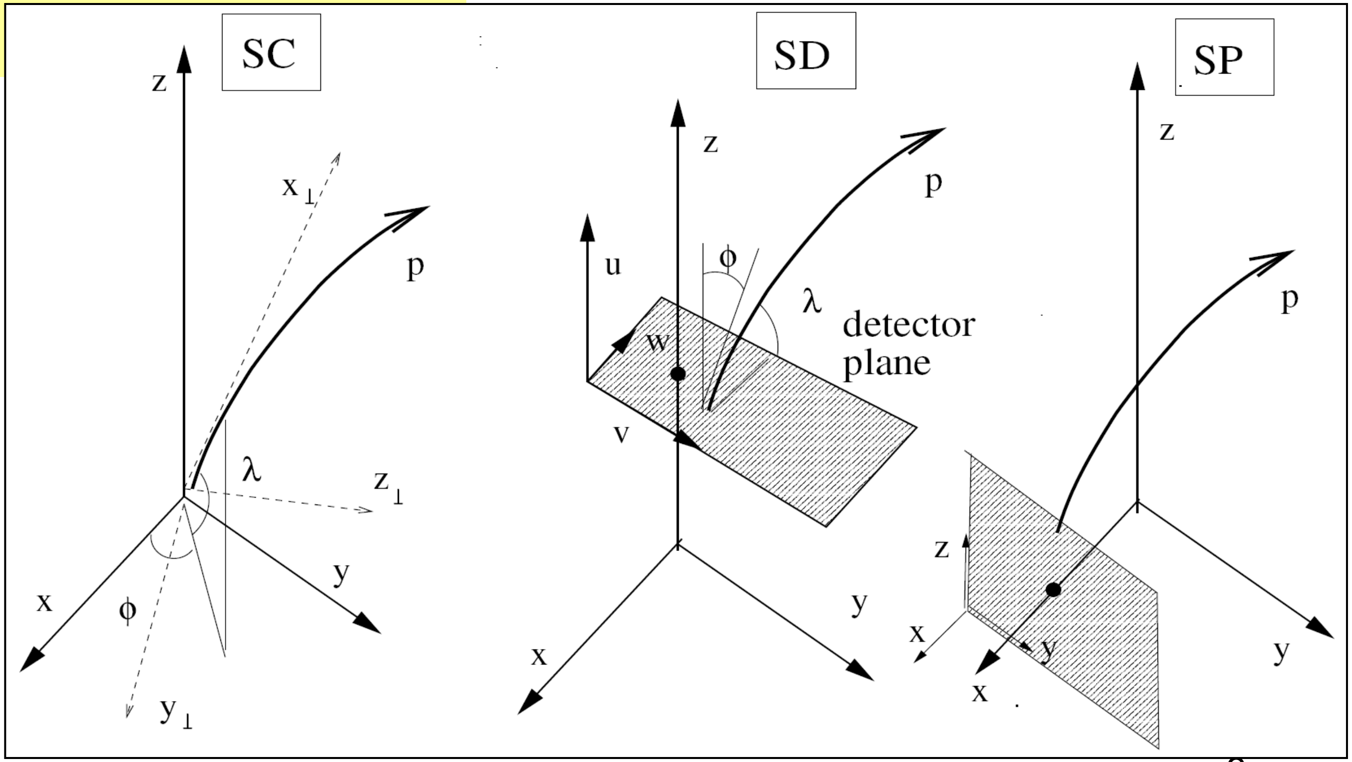
Two main tasks:

- **Track propagation**: the same MC geometry banks are used.
- **Error propagation**:
- from one point to another one
- In the same point between different systems

MARS → SC

$$\begin{pmatrix} x_\perp \\ y_\perp \\ z_\perp \end{pmatrix} = \begin{pmatrix} \cos\lambda\cos\varphi & \cos\lambda\sin\varphi & \sin\lambda \\ -\sin\varphi & \cos\varphi & 0 \\ -\sin\lambda\cos\varphi & -\sin\lambda\sin\varphi & \cos\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

GEANE = tracking with the geometry of GEANT3 + a lot of mathematics for the transport matrix calculation



Track propagation

Tracking:

$$e_j[k_i] = \boldsymbol{G}[k_i] \ ,$$

$\boldsymbol{G}$ is the software part that calculates the trajectory taking into account magnetic field and energy loss.

Error propagation:

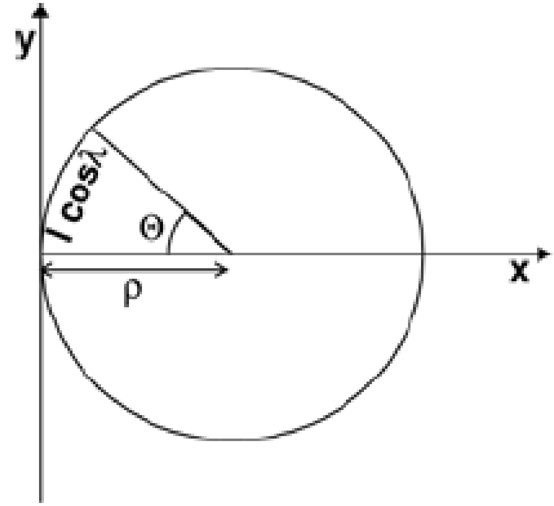
If $\boldsymbol{\sigma}[k_i]$ is the covariance matrix on the prediction k_i, the error on the extrapolated point e_i is given by the standard error propagation:

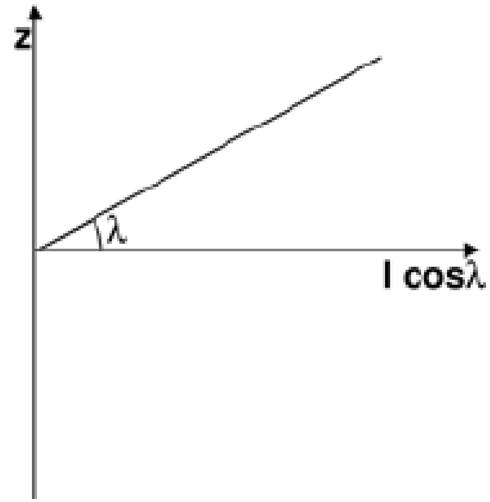
$$\boldsymbol{\sigma}[e_j] = \boldsymbol{T}_{ij}\boldsymbol{\sigma}[k_i]\boldsymbol{T}^T_{ij} + \boldsymbol{W}^{-1}_{ij} \qquad T_{ij}(l_2, l_1) = \frac{\partial e^i(l_2)}{\partial e^j(l_1)} \ ,$$

$\boldsymbol{T}_{ij}$ is the transport (derivative or gradient) matrix

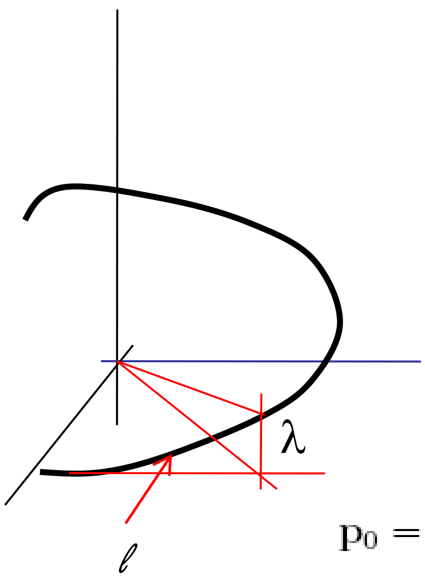
$\boldsymbol{W}_{ij}$ contains the errors (fluctuations) due to multiple scattering and energy loss. The calculation of this ma- [illegible] terribly complicated, so that usually people search for already existing and reliable products.

Track propagation II





a piece of helix
field along z-axis
M(s) is the position vector



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix} - \begin{pmatrix} \rho\cos\Theta \\ \rho\sin\Theta \end{pmatrix} \text{ and } \begin{cases} z &= l\cdot\cos\lambda\cdot\tan\lambda \\ \Theta\cdot\rho &= l\cdot\cos\lambda \end{cases}$$

it results:

$$\Theta\cdot\rho = l\cdot\cos\lambda$$

$$M(\Theta) = \rho\begin{pmatrix} 1-\cos\Theta \\ \sin\Theta \\ \Theta\tan\lambda \end{pmatrix}$$

$$p_0 = pT_0 = p\begin{pmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{pmatrix}$$

$$\frac{\partial M(l)}{\partial l} = T(l) = \cos\lambda\begin{pmatrix} \sin\Theta \\ \cos\Theta \\ \tan\lambda \end{pmatrix}$$

Track propagation III

Now the tracking can be performed, with the unique assumption for the field to be constant within one step, so that for an arbitrary magnetic field the track can be written as a series of helix pieces (one for each step). To perform the tracking let' s define an orthogonal right-handed triplet of axes (n_i, b_i, h_i):

$$
\begin{aligned}
\mathbf{h}_i &= \frac{\mathbf{H}_i}{|\mathbf{H}_i|} \\
\mathbf{n}_i &= \frac{\mathbf{T}_i \times \mathbf{h}_i}{|\mathbf{T}_i \times \mathbf{h}_i|} \\
\mathbf{b}_i &= \mathbf{h}_i \times \mathbf{n}_i
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}_{i+1} &= \mathbf{M}_i + \rho[(1 - \cos\Theta_i) \cdot \mathbf{n}_i + \sin\Theta_i \cdot \mathbf{b}_i + \Theta_i \tan\lambda_i \cdot \mathbf{h}_i] \\
\mathbf{T}_{i+1} &= \cos\lambda_i[\sin\Theta_i \cdot \mathbf{n}_i + \cos\Theta_i \cdot \mathbf{b}_i + \tan\lambda_i \cdot \mathbf{h}_i]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T} &= \frac{\mathbf{p}}{p} \\
\mathbf{N} &= \frac{\mathbf{H} \times \mathbf{T}}{|\mathbf{H} \times \mathbf{T}|} \\
\mathbf{R} &= \mathbf{T} \times \mathbf{N}
\end{aligned}
$$

The matrix to change from (n_i, b_i, h_i) to (N_i, R_i, T_i) is

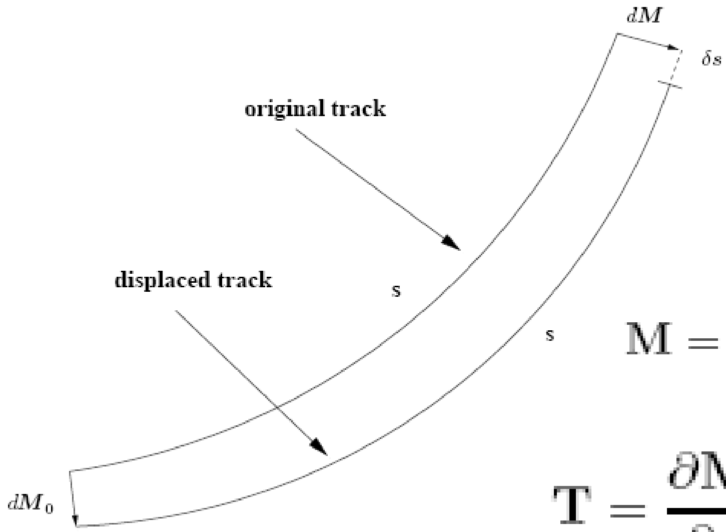
$$
\begin{pmatrix} N \\ R \\ T \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\sin\lambda & \cos\lambda \\ 0 & \cos\lambda & \sin\lambda \end{pmatrix} \begin{pmatrix} n \\ b \\ h \end{pmatrix}
$$

The helix can then be parametrized as follows ([7] and [9]):

$$
\mathbf{M} = \mathbf{M}_0 + \frac{\gamma}{Q}(\theta - \sin\theta) \cdot \mathbf{H} + \frac{\sin\theta}{Q} \cdot \mathbf{T}_0 + \frac{\alpha}{Q}(1 - \cos\theta) \cdot \mathbf{N}_0
$$

Track propagation IV

Strandlie & Wittek, CMS note, 2006/001



$$\mathbf{M} = \mathbf{M}_0 + \frac{\gamma}{Q}(\theta - \sin\theta)\cdot\mathbf{H} + \frac{\sin\theta}{Q}\cdot\mathbf{T}_0 + \frac{\alpha}{Q}(1-\cos\theta)\cdot\mathbf{N}_0,$$

$$\mathbf{T} = \frac{\partial\mathbf{M}}{\partial s} = \gamma(1-\cos\theta)\cdot\mathbf{H} + \cos\theta\cdot\mathbf{T}_0 + \alpha\sin\theta\cdot\mathbf{N}_0.$$

with $\mathbf{M}$ being the position vector of the point on the helix at path length s from the reference point $\mathbf{M}_0$ (at $s = 0$), $\mathbf{H} = \mathbf{B}/|\mathbf{B}|$ being a normalized magnetic field vector, $\mathbf{T} = \mathbf{p}/|\mathbf{p}|$ being a normalized tangent vector to the track, $\mathbf{N} = (\mathbf{H}\times\mathbf{T})/\alpha$ with $\alpha = |\mathbf{H}\times\mathbf{T}|$, $\gamma = \mathbf{H}\cdot\mathbf{T}$, $Q = -|\mathbf{B}|\,q/p$ with $p = |\mathbf{p}|$ being the absolute value of the 3-momentum vector, $q = \pm 1$ denoting the charge of the particle, and $\theta = Q\cdot s$. The numerical value of $|\mathbf{B}|$ is

$$\begin{aligned} d\mathbf{M} &= \frac{\partial\mathbf{M}}{\partial\mathbf{M}_0}\cdot d\mathbf{M}_0 + \frac{\partial\mathbf{M}}{\partial\mathbf{T}_0}\cdot d\mathbf{T}_0 + \frac{\partial\mathbf{M}}{\partial(q/p_0)}\cdot\delta(q/p_0) + \frac{\partial\mathbf{M}}{\partial s}\cdot\delta s, \\ d\mathbf{T} &= \frac{\partial\mathbf{T}}{\partial\mathbf{T}_0}\cdot d\mathbf{T}_0 + \frac{\partial\mathbf{T}}{\partial(q/p_0)}\cdot\delta(q/p_0) + \frac{\partial\mathbf{T}}{\partial s}\cdot\delta s, \end{aligned}$$

The first task: track propagation

$$\frac{\partial(q/p)}{\partial(q/p_0)} = 1,$$

$$\frac{\partial\lambda}{\partial(q/p_0)} = -\alpha Q\cdot\left(\frac{q}{p}\right)^{-1}\cdot(\mathbf{N}\cdot\mathbf{V})\cdot[\mathbf{T}\cdot(\mathbf{M}_0-\mathbf{M})],$$

$$\begin{aligned}\frac{\partial\lambda}{\partial\lambda_0} = {} & \cos\theta\cdot(\mathbf{V}_0\cdot\mathbf{V})+\sin\theta\cdot((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{V}) \\ & +(1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{V}) \\ & +\alpha(\mathbf{N}\cdot\mathbf{V})[-\sin\theta(\mathbf{V}_0\cdot\mathbf{T})+\alpha(1-\cos\theta)(\mathbf{V}_0\cdot\mathbf{N}) \\ & -(\theta-\sin\theta)(\mathbf{H}\cdot\mathbf{T})(\mathbf{H}\cdot\mathbf{V}_0)],\end{aligned}$$

$$\begin{aligned}\frac{\partial\lambda}{\partial\phi_0} = {} & \cos\lambda_0\{\cos\theta\cdot(\mathbf{U}_0\cdot\mathbf{V})+\sin\theta\cdot((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{V}) \\ & +(1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{V}) \\ & +\alpha(\mathbf{N}\cdot\mathbf{V})[-\sin\theta(\mathbf{U}_0\cdot\mathbf{T})+\alpha(1-\cos\theta)(\mathbf{U}_0\cdot\mathbf{N}) \\ & -(\theta-\sin\theta)(\mathbf{H}\cdot\mathbf{T})(\mathbf{H}\cdot\mathbf{U}_0)]\},\end{aligned}$$

$$\frac{\partial\lambda}{\partial x_{\perp 0}} = -\alpha Q(\mathbf{N}\cdot\mathbf{V})(\mathbf{U}_0\cdot\mathbf{T}),$$

$$\frac{\partial\lambda}{\partial y_{\perp 0}} = -\alpha Q(\mathbf{N}\cdot\mathbf{V})(\mathbf{V}_0\cdot\mathbf{T}),$$

$$\frac{\partial\phi}{\partial(q/p_0)} = -\frac{\alpha Q}{\cos\lambda}\cdot\left(\frac{q}{p}\right)^{-1}\cdot(\mathbf{N}\cdot\mathbf{U})\cdot[\mathbf{T}\cdot(\mathbf{M}_0-\mathbf{M})],$$

$$\begin{aligned}\frac{\partial\phi}{\partial\lambda_0} = {} & \frac{1}{\cos\lambda}\{\cos\theta\cdot(\mathbf{V}_0\cdot\mathbf{U})+\sin\theta\cdot((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{U}) \\ & +(1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{U}) \\ & +\alpha(\mathbf{N}\cdot\mathbf{U})[-\sin\theta(\mathbf{V}_0\cdot\mathbf{T})+\alpha(1-\cos\theta)(\mathbf{V}_0\cdot\mathbf{N}) \\ & -(\theta-\sin\theta)(\mathbf{H}\cdot\mathbf{T})(\mathbf{H}\cdot\mathbf{V}_0)]\},\end{aligned}$$

$$\begin{aligned}\frac{\partial\phi}{\partial\phi_0} = {} & \frac{\cos\lambda_0}{\cos\lambda}\{\cos\theta\cdot(\mathbf{U}_0\cdot\mathbf{U})+\sin\theta\cdot((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{U}) \\ & +(1-\cos\theta)\cdot(\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{U}) \\ & +\alpha(\mathbf{N}\cdot\mathbf{U})[-\sin\theta(\mathbf{U}_0\cdot\mathbf{T})+\alpha(1-\cos\theta)(\mathbf{U}_0\cdot\mathbf{N}) \\ & -(\theta-\sin\theta)(\mathbf{H}\cdot\mathbf{T})(\mathbf{H}\cdot\mathbf{U}_0)]\},\end{aligned}$$

$$\frac{\partial\phi}{\partial x_{\perp 0}} = -\frac{\alpha Q}{\cos\lambda}(\mathbf{N}\cdot\mathbf{U})(\mathbf{U}_0\cdot\mathbf{T}),$$

$$\frac{\partial\phi}{\partial y_{\perp 0}} = -\frac{\alpha Q}{\cos\lambda}(\mathbf{N}\cdot\mathbf{U})(\mathbf{V}_0\cdot\mathbf{T}),$$

$$\frac{\partial x_\perp}{\partial(q/p_0)} = \left(\frac{q}{p}\right)^{-1}[\mathbf{U}\cdot(\mathbf{M}_0-\mathbf{M})],$$

$$\begin{aligned}\frac{\partial x_\perp}{\partial\lambda_0} = {} & \frac{\sin\theta}{Q}(\mathbf{V}_0\cdot\mathbf{U})+\frac{1-\cos\theta}{Q}((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{U}) \\ & +\frac{\theta-\sin\theta}{Q}(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{U}),\end{aligned}$$

$$\begin{aligned}\frac{\partial x_\perp}{\partial\phi_0} = {} & \cos\lambda_0\left\{\frac{\sin\theta}{Q}(\mathbf{U}_0\cdot\mathbf{U})+\frac{1-\cos\theta}{Q}((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{U})\right. \\ & \left.+\frac{\theta-\sin\theta}{Q}(\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{U})\right\},\end{aligned} \quad (63)$$

$$\frac{\partial x_\perp}{\partial x_{\perp 0}} = \mathbf{U}_0\cdot\mathbf{U}, \quad (64)$$

$$\frac{\partial x_\perp}{\partial y_{\perp 0}} = \mathbf{V}_0\cdot\mathbf{U}, \quad (65)$$

$$\frac{\partial y_\perp}{\partial(q/p_0)} = \left(\frac{q}{p}\right)^{-1}[\mathbf{V}\cdot(\mathbf{M}_0-\mathbf{M})], \quad (66)$$

$$\begin{aligned}\frac{\partial y_\perp}{\partial\lambda_0} = {} & \frac{\sin\theta}{Q}(\mathbf{V}_0\cdot\mathbf{V})+\frac{1-\cos\theta}{Q}((\mathbf{H}\times\mathbf{V}_0)\cdot\mathbf{V}) \\ & +\frac{\theta-\sin\theta}{Q}(\mathbf{H}\cdot\mathbf{V}_0)\cdot(\mathbf{H}\cdot\mathbf{V}),\end{aligned} \quad (67)$$

$$\begin{aligned}\frac{\partial y_\perp}{\partial\phi_0} = {} & \cos\lambda_0\left\{\frac{\sin\theta}{Q}(\mathbf{U}_0\cdot\mathbf{V})+\frac{1-\cos\theta}{Q}((\mathbf{H}\times\mathbf{U}_0)\cdot\mathbf{V})\right. \\ & \left.+\frac{\theta-\sin\theta}{Q}(\mathbf{H}\cdot\mathbf{U}_0)\cdot(\mathbf{H}\cdot\mathbf{V})\right\},\end{aligned} \quad (68)$$

$$\frac{\partial y_\perp}{\partial x_{\perp 0}} = \mathbf{U}_0\cdot\mathbf{V}, \quad (69)$$

$$\frac{\partial y_\perp}{\partial y_{\perp 0}} = \mathbf{V}_0\cdot\mathbf{V}, \quad (70)$$

ns numerically stable at small values of θ are given by

$$\begin{aligned}\frac{\partial x_\perp}{\partial(q/p_0)} = {} & -\frac{1}{2}|\mathbf{B}|s^2\cdot(\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{U}+\frac{1}{3}|\mathbf{B}|^2s^3\cdot\frac{q}{p}\cdot(\gamma\mathbf{H}-\mathbf{T}_0)\cdot\mathbf{U} \\ & +\frac{1}{8}|\mathbf{B}|^3s^4\cdot\left(\frac{q}{p}\right)^2\cdot(\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{U},\end{aligned} \quad (71)$$

$$\begin{aligned}\frac{\partial y_\perp}{\partial(q/p_0)} = {} & -\frac{1}{2}|\mathbf{B}|s^2\cdot(\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{V}+\frac{1}{3}|\mathbf{B}|^2s^3\cdot\frac{q}{p}\cdot(\gamma\mathbf{H}-\mathbf{T}_0)\cdot\mathbf{V} \\ & +\frac{1}{8}|\mathbf{B}|^3s^4\cdot\left(\frac{q}{p}\right)^2\cdot(\mathbf{H}\times\mathbf{T}_0)\cdot\mathbf{V}.\end{aligned} \quad (72)$$

$$\boldsymbol{\sigma}^2(l_2) = \boldsymbol{T}(l_2, l_1)\,\boldsymbol{\sigma}^2(l_1)\,\boldsymbol{T}^T(l_2, l_1) + \boldsymbol{W}^{-1}(l_1)$$

The jacobian transports the erros from one step to another

Here, at each step, multiple scattering and energy loss effects have to be added

Multiple scattering

On the quantitative modelling of core and tails of multiple scattering by Gaussian mixtures

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Received 25 April 2000; accepted 30 May 2000

$$\frac{d\sigma}{d\theta} = 2\pi\left(\frac{2Ze^2}{pv}\right)^2 \frac{\theta}{(\theta^2 + \theta_{\min}^2)^2}. \rightarrow f(\theta) = \frac{k\theta}{(\theta^2 + \theta_{\min}^2)^2} I_{[0,\theta_{\max}]}(\theta)$$

There is no simple closed form for the cumulative distribution function of the projected scattering angle. For the simulation one therefore has to go back to the scattering angle θ in space, which can be generated by inverting its cumulative distribution function:

$$\theta = ab\sqrt{\frac{1-u}{ub^2 + a^2}} \tag{26}$$

where u is a stochastic variable with a uniform distribution in the interval [0,1]. If φ is uniform in

Multiple scattering

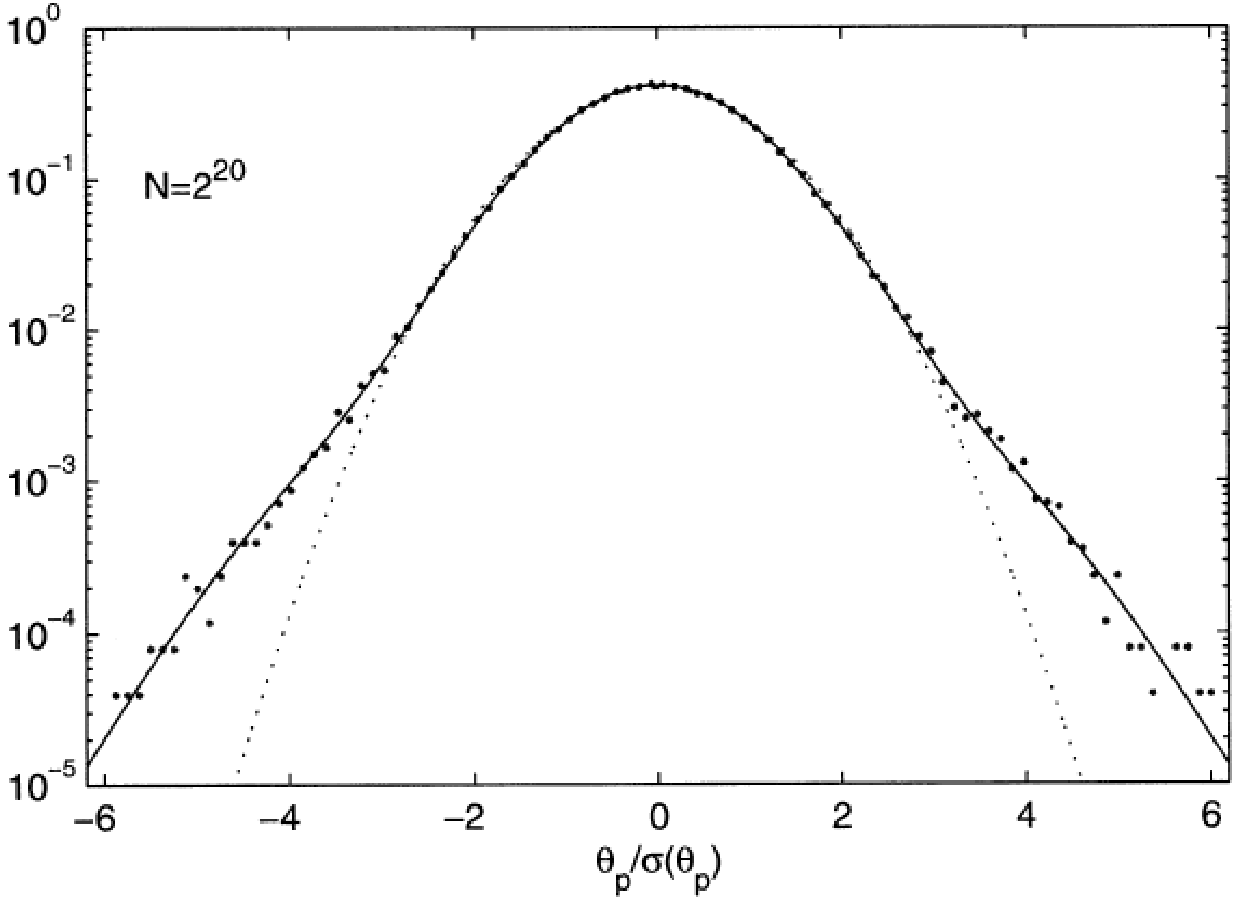


Fig. 3. The density of the projected multiple scattering angle in carbon, in standard measure, for $N = 2^{10}$ (top) and $N = 2^{20}$ (bottom). The dots are the frequencies of a simulated sample obtained by summing over single scatters. The dotted line is the density of a standard Gaussian.

Multiple scattering

Molière's final solution $f_M(\theta)\theta\,d\theta$ of the transport equation is given in space, using the transformation

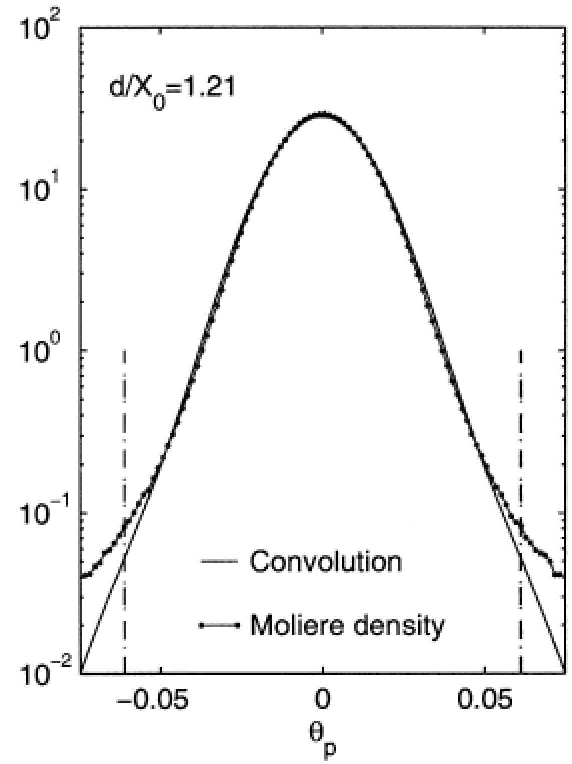
$$f(\theta)d\theta = f_M(\theta)\,d(\cos\theta)\,d\varphi/2\pi \qquad (49)$$

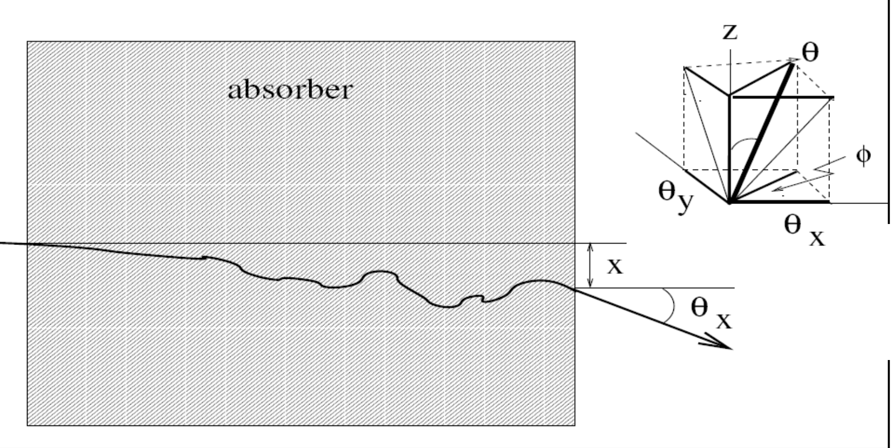
and the approximation $|d(\cos\theta)| = \sin\theta\,d\theta \approx \theta\,d\theta$. In his solution the function $f_M(\theta)$ is approximated by

$$f_M(\theta) \approx \frac{1}{2\,\theta_M^2}\left[f^{(0)}(\theta') + \frac{f^{(1)}(\theta')}{B} + \frac{f^{(2)}(\theta')}{B^2}\right] \qquad (50)$$

where θ_M is the characteristic multiple scattering angle of the target, $\theta' = \theta/(\sqrt{2}\theta_M)$ is the reduced angle, and B is related to the logarithm of the effective number of collisions in the target. The functions $f^{(k)}$ are given by

$$f^{(k)}(\theta') = \frac{1}{n!}\int_0^\infty y\,J_0(\theta' y)\,e^{-y^2/4}\left(\frac{y^2}{4}\ln\frac{y^2}{4}\right)^k dy \qquad (51)$$





Multiple scattering

$$< \theta_p^2 > , \quad < x^2 >= \frac{< \theta_p^2 > d^2}{3} , \quad < x, \theta_p >= \frac{< \theta_p^2 > d}{2}$$

$$p(x, \theta_p; d) = \frac{2\sqrt{3}}{\pi} \frac{1}{< \theta_p^2 > d^2} \exp\left[-\frac{4}{< \theta_p^2 >} \left(\frac{\theta_p^2}{d} - \frac{3x\theta_p}{d^2} + \frac{3x^2}{d^3} \right) \right]$$

$$< \theta_p^2 >= \frac{(0.0136)^2 \, d}{p^2 \, \beta^2 X_0} \left[1 + 0.038 \ln\left(\frac{d}{X_0} \right) \right]^2$$

PDG: wrong

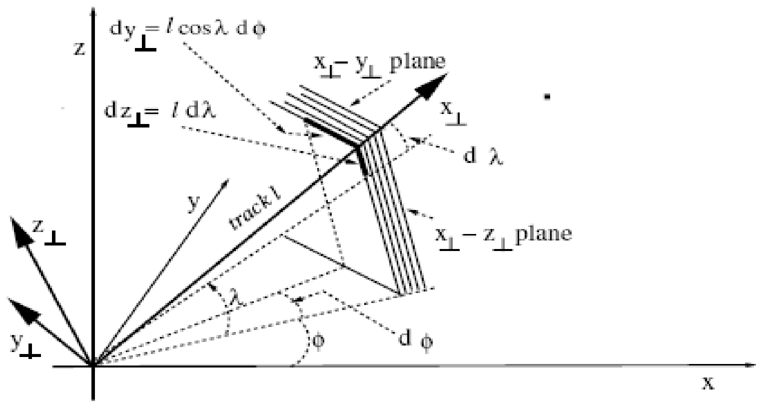
$$< \theta_p^2 >= \frac{184.96 \cdot 10^{-6}}{p^2} \frac{d}{\beta^2 X_0} .$$

GEANE

$$< \theta_p^2 >= \frac{225 \cdot 10^{-6}}{p^2} \frac{d}{\beta^2 X_s} , \qquad X_s = X_0 \frac{Z+1}{Z} \frac{\ln(287 \, Z^{-1/2})}{\ln(159 \, Z^{-1/3})}$$

R. Frühwirth and M. Regler, Nucl. Instr. and Meth. A456(2001)369

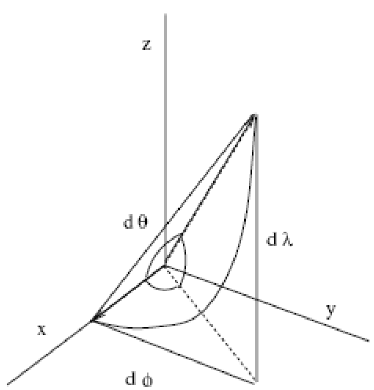
$$\boldsymbol{\sigma}^2(l_2) = \boldsymbol{T}(l_2, l_1)\,\boldsymbol{\sigma}^2(l_1)\,\boldsymbol{T}^T(l_2, l_1) + \boldsymbol{W}^{-1}(l_1)$$



$$dz_\perp = l\, d\lambda\ , \qquad dy_\perp = l \cos\lambda d\phi$$

$$\lambda \equiv -\theta_z\ , \quad \phi \equiv \frac{\theta_y}{\cos\lambda}\ , \quad y_\perp \equiv y\ , \quad z_\perp \equiv z$$

$$< t_i, t_j > = \sum_{lm} \frac{\partial t_i}{\partial s_l} \frac{\partial t_j}{\partial s_m} < s_l, s_m >$$



	1/p	λ	φ	$y_\perp$	$z_\perp$
1 /p	0	0	0	0	0
λ	0	$<\theta_p^2>$	0	0	$-\frac{<\theta_p^2>\ \mathrm{d}l}{2}$
φ	0	0	$\frac{<\theta_p^2>}{\cos^2\lambda}$	$\frac{<\theta_p^2>\ \mathrm{d}l}{(2\cos\lambda)}$	0
$y_\perp$	0	0	$\frac{<\theta_p^2>\ \mathrm{d}l}{(2\cos\lambda)}$	$\frac{<\theta_p^2>\ (\mathrm{d}l)^2}{3}$	
$z_\perp$	0	$-\frac{<\theta_p^2>\ \mathrm{d}l}{2}$	0	0	$\frac{<\theta_p^2>\ (\mathrm{d}l)^2}{3}$

Energy loss

The fluctuations in ionization for one particle of charge z, mass m, velocity β, are characterized by the parameter κ,

$$\kappa = \frac{\xi}{E_{\max}} \,, \tag{60}$$

which is proportional to the ratio of mean energy loss to the maximum allowed energy transfer $E_{\max}$ in a single collision with an atomic electron:

$$E_{\max} = \frac{2m_e\beta^2\gamma^2}{1 + 2\gamma m_e/m + (m_e/m)^2} \,, \tag{61}$$

where $\gamma = 1/\sqrt{1-\beta^2} = E/m$ and m_e is the electron mass. The parameter ξ comes from the Rutherford scattering cross section and is defined as [11]:

$$\xi = 153.4 \frac{z^2 Z}{\beta^2 A} \rho d \quad (\mathrm{keV}) \,, \tag{62}$$

where ρ, d, Z and A are the density (g/cm^3), thickness, atomic and mass number of the medium.

Average energy loss

$$\left\langle \frac{dE}{dx} \right\rangle = -4\pi N_A r_e^2 m_e c^2 z^2 \frac{Z}{A}\frac{1}{\beta^2}\left[\tfrac{1}{2}\ln\frac{2 m_e c^2 \gamma^2 \beta^2}{I^2} T^{max} - \beta^2 - \frac{\delta}{2}\right]$$

BETHE-BLOCH

Fluctuations in energy loss

$$k = \frac{\xi}{E_{max}} = \frac{\textbf{average energy loss}}{\textbf{max energy loss in a single collision}}$$

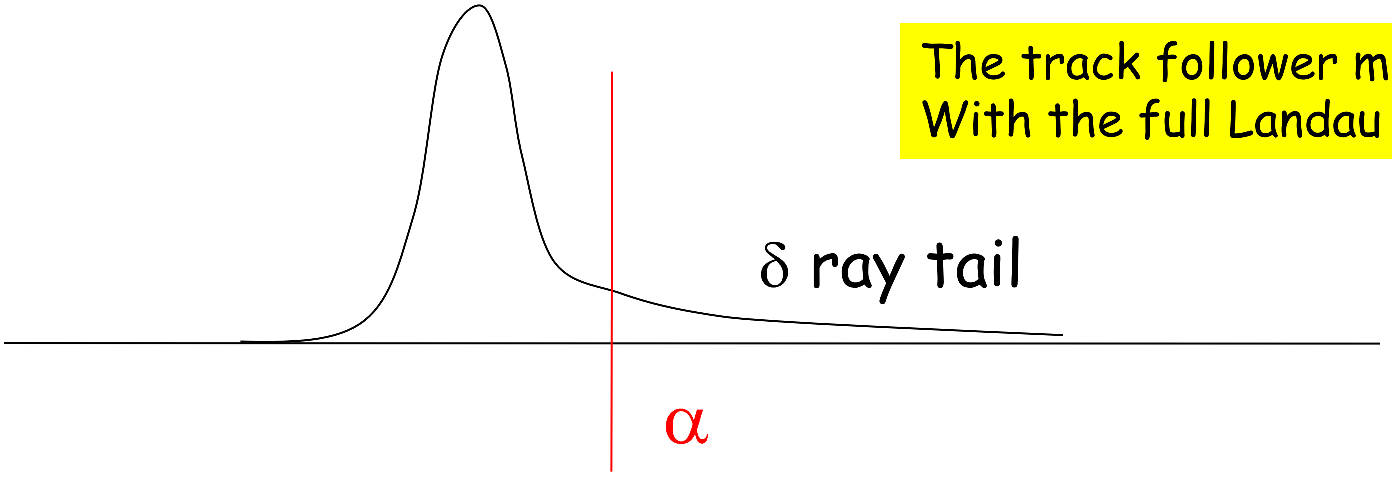
$k > 10$	Gaussian	$\sigma^2(E) = \xi E_{max}\left(1 - \frac{\beta^2}{2}\right) \Rightarrow \sigma^2\left(\frac{1}{p}\right) \Rightarrow \sigma_{11}^2$
$0.01 < k < 10$	Vavilov	
$k < 0.01;\ N_c > 50$	Landau	μ, σ are infinite !!
$k < 0.01;\ N_c < 50$	Sub-Landau	σ is too large!!

Gauss and Vavilov: no problems for the track follower

GEANE

$$\sigma^2(E) = \xi\, E_{max}\left(1-\frac{\beta^2}{2}\right) \Rightarrow \sigma^2\left(\frac{1}{p}\right) \Rightarrow \sigma^2_{11}$$

Landau: problematic distribution



The track follower must be compare
With the full Landau sampling

Sub-Landau: what distribution?

Gauss and Vavilov: no problems for the track follower

$$\sigma^2(E) = \frac{\xi^2}{\kappa}(1-\beta^2/2) = \xi\, E_{\rm max}(1-\beta^2/2)\ . \qquad (63)$$

Taking into account the energy-momentum equation

$$E^2 = p^2 + m^2 \quad \rightarrow \quad \frac{\mathrm{dp}}{\mathrm{dE}} = \frac{E}{p} = \frac{1}{\beta}\ ,$$

and the error transformation

$$\sigma^2(1/p) = \left[\frac{\mathrm{d}}{\mathrm{dp}}\left(\frac{1}{p}\right)\right]^2 \sigma^2(p) = \frac{1}{p^4}\,\sigma^2(p) = \frac{E^2}{p^6}\,\sigma^2(E)$$

GEANE and GEANT4E contain only this

Improvements

- New error calculation in energy loss for heavy particles
- New error calculation for bremsstrahlung

Truncated Landau:

λ_{max}	α	Mean	σ_α
11.1	0.90	1.61	2.83
22.4	0.95	2.40	4.23
110.0	0.99	4.19	10.16
200.0	0.995	4.82	13.88
256.0	0.996	5.08	15.76
339.0	0.997	5.37	18.19
507.0	0.998	5.78	22.33
1007.0	0.999	6.48	31.59

Table 1: Result of the integration $\alpha = \int_{\lambda_{min}}^{\lambda_{max}} f(\lambda)\, d\lambda$ of the Landau distribution from $\lambda_{min} \simeq -3.5$ to λ_{max} of the table. The mean and the standard deviation of the truncated distribution are also shown. For this distribution. the full mean and the variance are infinite, only the cumulative can be calculated

Solution (GEANT3 & GEANT4): truncation of the distribution tail to have as a mean the average dE/dx

$$\lambda_{max} = 0.60715 + 1.1934\,\langle\lambda\rangle + (0.67794 + 0.052382\,\langle\lambda\rangle)\exp(0.94753 + 0.74442\,\langle\lambda\rangle)$$

Original GEANE

GEANE for PANDA modified with the the α-tail

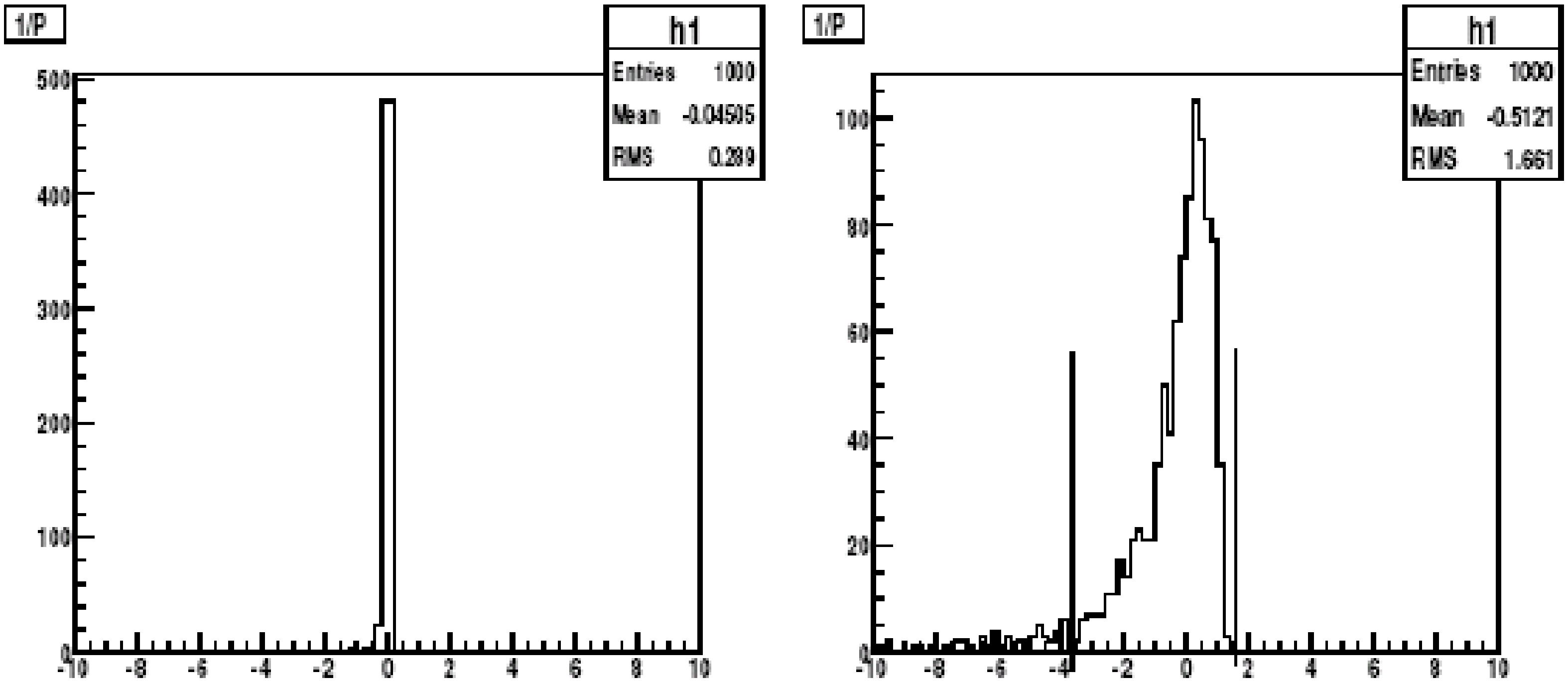
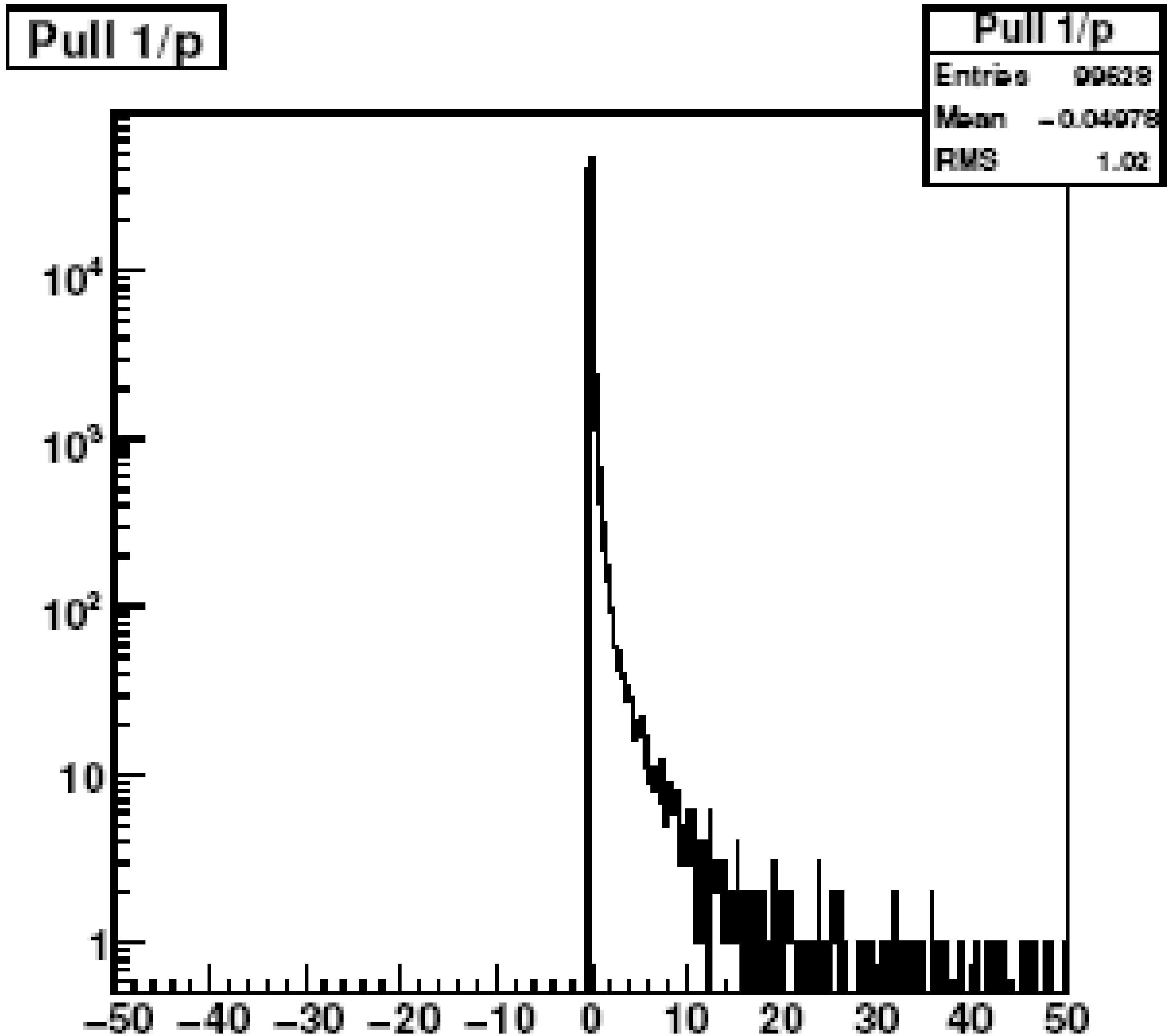


Figure 10: Pull distribution $\Delta(1/p)/\sigma$ for 1 GeV muons after passing through the PANDA straw tube detector. Left: Standard GEANE result (RMS$\simeq$ 0.3 in the displayed window); right: result after the modification with $\alpha = 0.995$ (see the text). The region between the vertical lines has RMS= 1.03.


Pull 1/p
Pull 1/p
Entries 99628
Mean −0.04978
RMS 1.02
10^4
10^3
10^2
10
1
−50 −40 −30 −20 −10 0 10 20 30 40 50

Thin gaseous absorbers: The Urban distribution

- excitation macroscopic cross sections Σ_1 and Σ_2:

$$\Sigma_i = C \frac{f_i}{E_i} \frac{\ln(2m\beta^2\gamma^2/e_i) - \beta^2}{\ln(2m\beta^2\gamma^2/I) - \beta^2}(1-r) \ , \quad i = 1,2$$

$$I = 16Z^{0.9} \text{ (eV)} \ , \quad f_2 = \begin{cases} 0 \text{ if} Z \leq 2 \\ 2/Z \text{ if} Z > 2 \end{cases} \ , \quad f_1 = 1 - f_2$$

$$e_2 = 10Z^2 \text{ (eV)} \ , \quad e_1 = \left(\frac{I}{e_2^{f_2}}\right)^{1/f_1} \ , \quad \boxed{r = 0.4} , \quad C = \frac{E_{\text{med}}}{\Delta x} \ ,$$

and $E_{\text{med}} \equiv (\text{d}E/\text{d}x) \cdot \Delta x$ is the energy lost in the absorber of thickness Δx;

- ionization macroscopic cross section Σ_3:

$$\Sigma_3 = C \frac{E_{\text{max}}}{I(E_{\text{max}} + I)\ln((E_{\text{max}} + I)/I)} r$$

- number of total collisions N_c:

$$N_c = (\Sigma_1 + \Sigma_2 + \Sigma_3)\Delta x = N_1 + N_2 + N_3 \ . \qquad (8)$$

$$E = (\Sigma_1 e_1 + \Sigma_2 e_2 + \Sigma_3 E_3)\Delta x = N_1 e_1 + N_2 e_2 + N_3 E_3 \ , \tag{9}$$

where e_1 and e_2 are the two fixed excitation energies of the model and E_3 is the energy lost by δ-electron emission. This is a stochastic quantity that follows approximately the distribution [?]:

δ-ray tail

$$E_3 \sim g(E) \text{ where } g(E) = \frac{I(E_{\max} + I)}{E_{\max}} \frac{1}{E^2} \ , \quad I < E < E_{\max} + I \ . \tag{10}$$

In GEANT3 and GEANT4 the energy E is obtained by eq. (9) by sampling N_1, N_2 and N_3 from the Poisson distribution and E_3 from $g(E)$.

Therefore, the sampling of the excitation energy is

$$E_e = N_1 e_1 + N_2 e_2 \ , \tag{11}$$

with E_1 and E_2 are consant and N_1, N_2 are sample from the Poisson distribution, whereas the delta ray ionization energy is sampled as:

$$E_i = \sum_{j=1}^{N_3} \frac{I}{1 - u(E_{\max}/(E_{\max} + I))} \ . \tag{12}$$

Truncation of the Urban tail of distribution

$$\frac{I(E_{\max}+I)}{E\max}\int_I^{E_\alpha}\frac{1}{E^2}\,\mathrm{d}E=\frac{(E_{\max}+I)}{E_{\max}}\frac{E_\alpha-I}{E_\alpha}=\alpha$$

$$\rightarrow E_\alpha=\frac{I}{1-\alpha\,E_{\max}/(E_{\max}+I)}$$

The mean and variance of the truncated distribution are:

$$\begin{aligned}\langle E_3\rangle &= \frac{I(E_{\max}+I)}{E_{\max}}\int_I^{E_\alpha}\frac{1}{E}\,\mathrm{d}E=\frac{I(E_{\max}+I)}{E_{\max}}\ln\left(\frac{E_\alpha}{I}\right)\ ,\\ \langle E_3^2\rangle &= \frac{I(E_{\max}+I)}{E_{\max}}\int_I^{E_\alpha}\mathrm{d}E=\frac{I(E_{\max}+I)}{E_{\max}}(E_\alpha-I)\ ,\\ \sigma_\alpha^2(E_3) &= <E_3^2>-<E_3>^2\ . \end{aligned} \qquad (13)$$

Then, the error propagation applied to eq. (9), where a random sum is present, where N_1, N_2, N_3 and E_3 are random variables, gives:

$$\sigma^2(E)=\langle N_1\rangle\,e_1^2+\langle N_2\rangle\,e_2^2+\langle N_3\rangle\,\langle E_3\rangle^2+\sigma^2[E_3]\,\langle N_3\rangle \qquad (14)$$

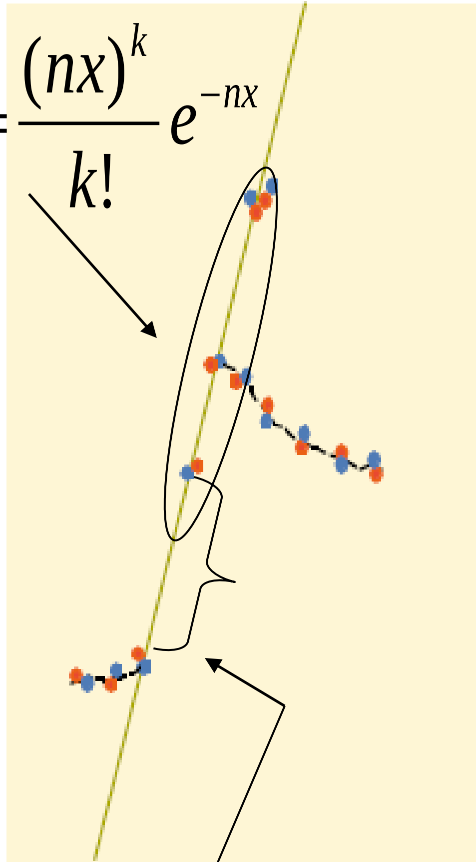
Is the Urban distribution a good model?

Comparison with an "exact" model
in the case of a thin gas layer

SECONDARY AND TOTAL IONIZATION

CLUSTERS AND DELTA ELECTRONS:

$$p(n)=\frac{(nx)^k}{k!}e^{-nx}$$



$$l(x)=n\,e^{-nx}$$

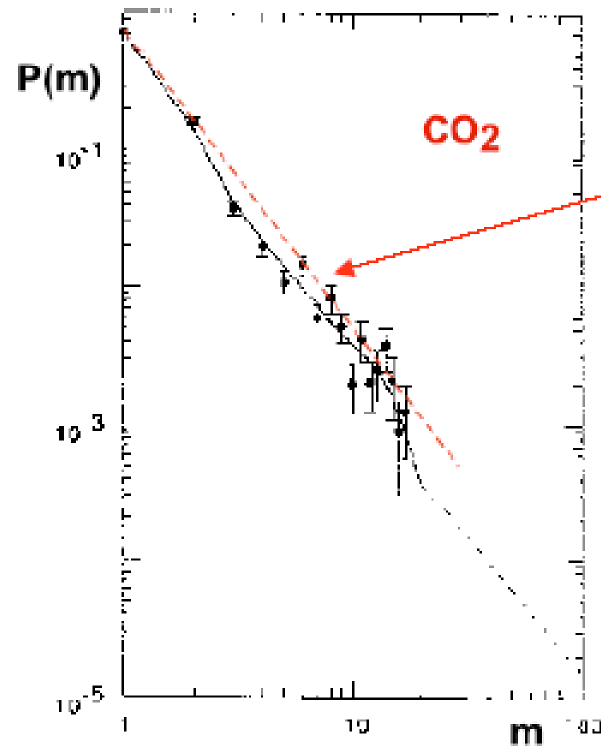
MIP particle:

	Argon	CO2
Cluster/cm	26	35

Effective cluster/cm: $n_{MIP}\, dE/dx / (dE/dx)_{min}$

N: total ion-electron pairs **N/n ~ 2.8**

CLUSTER SIZE DISTRIBUTION:



$$P(m) \sim \frac{W}{m^2}$$

H. Fischle et al, Nucl. Instr. and Meth. A301(1991)202

Urban model works well

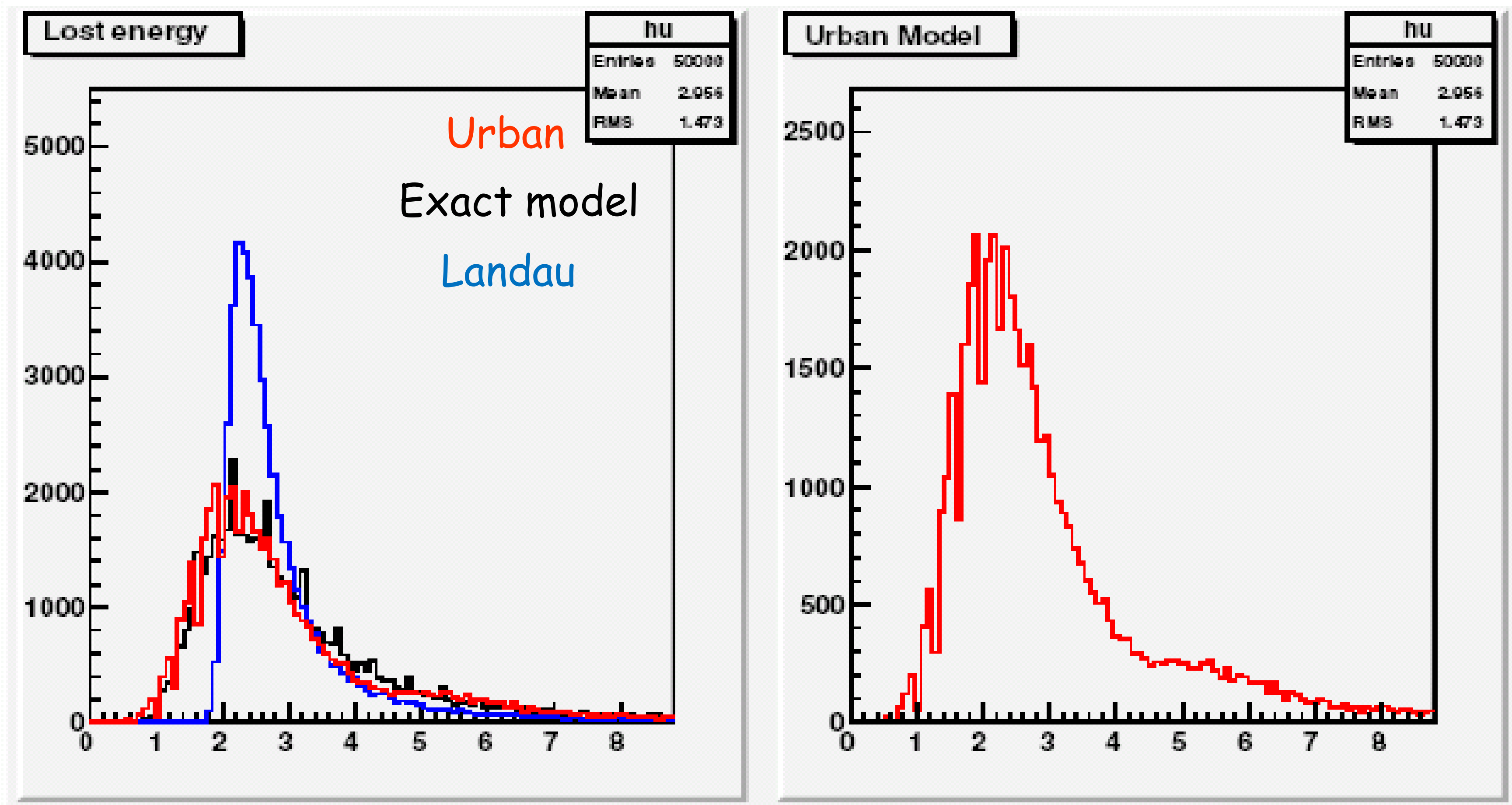


Figure 2: Urban and simulated distribution

1.5 cm of Ar/CO2 90/10 1.2 GeV pions

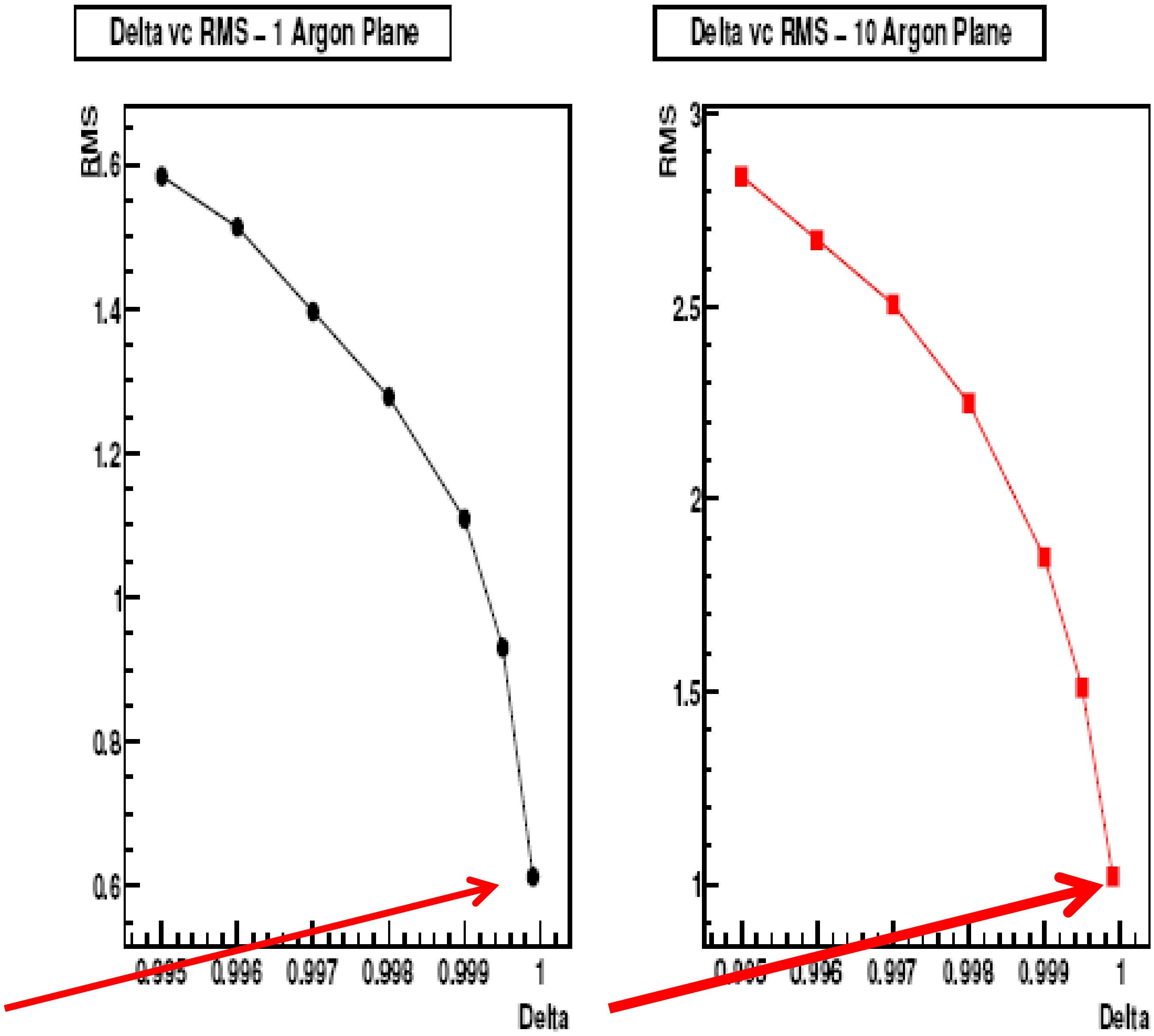
In summary, our method calculates the $1/p$ variance of eq. (5) with a variance $\sigma^2(E)$ due to the ionization energy loss calculated as follows:

a) for big and moderate absorbers when $\kappa > 0.01$, the variance $\sigma^2(E)$ is given by eq. (4) (old GEANE method);

b) for thin absorbers, $\kappa < 0.01$, when the number of collisions from eq. (10) is $N_c > 50$, $\sigma^2(E)$ is given by eq. (9);

c) for very thin absorbers, when $\kappa < 0.01$ and $N_c < 50$, the variance $\sigma^2(E)$ is given by eq. (17).

The matching between Urban and Landau is obtained for $\delta = 0.9999$


Delta vc RMS - 1 Argon Plane
RMS
Delta
Delta vc RMS - 10 Argon Plane
RMS
Delta

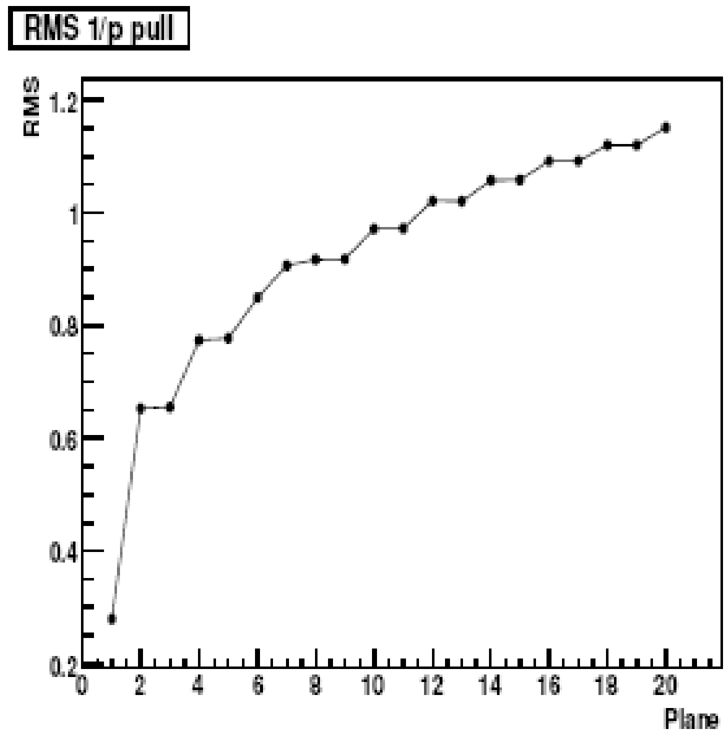


Figure 4: Values of the standard deviations of the $1/p$ pull variable with truncation parameter $\delta = 0.9999$ from eq. (15), as a function of the number of the traversed layers. The data refer to 1 Gev pions traversing layers formed by a 1 mm thick Al (Landau distribution) and a 1 cm thick Ar gas (Urban distribution) absorbers at NTP.

Bremsstrahlung

The radiative energy loss straggling distribution for the energy E of a particle of incident energy E_0 on an absorber of thickness x, was first deduced by Heitler [28], using an approximate expression for the bremsstrahlung cross section:

$$f(E) = \frac{1}{E_0 \Gamma(l)} \left(\ln \frac{E_0}{E} \right)^{l-1} , \quad l = \frac{x}{X_0 \ln 2} , \tag{18}$$

where X_0 is the radiation length of the absorber and Γ is the gamma function.

$$\langle E \rangle = E_0 \frac{1}{2^l} , \qquad \langle E^2 \rangle = E_0^2 \frac{1}{3^l}$$

$$\sigma^2[E] = \langle E^2 \rangle - \langle E \rangle^2 = E_0^2 \left(\frac{1}{3^l} - \frac{1}{4^l} \right) .$$

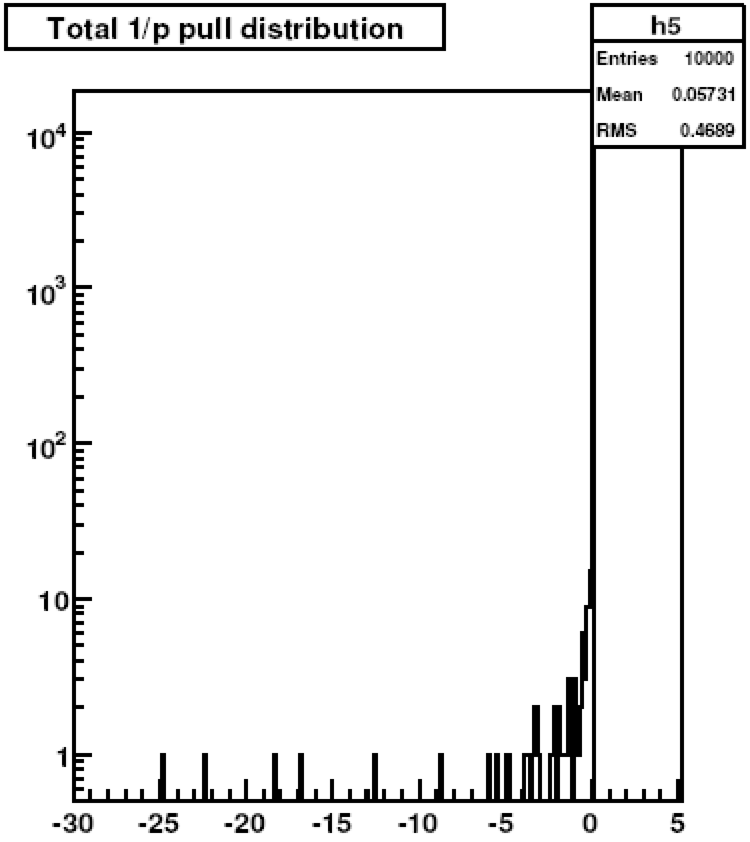
Bremsstrahlung

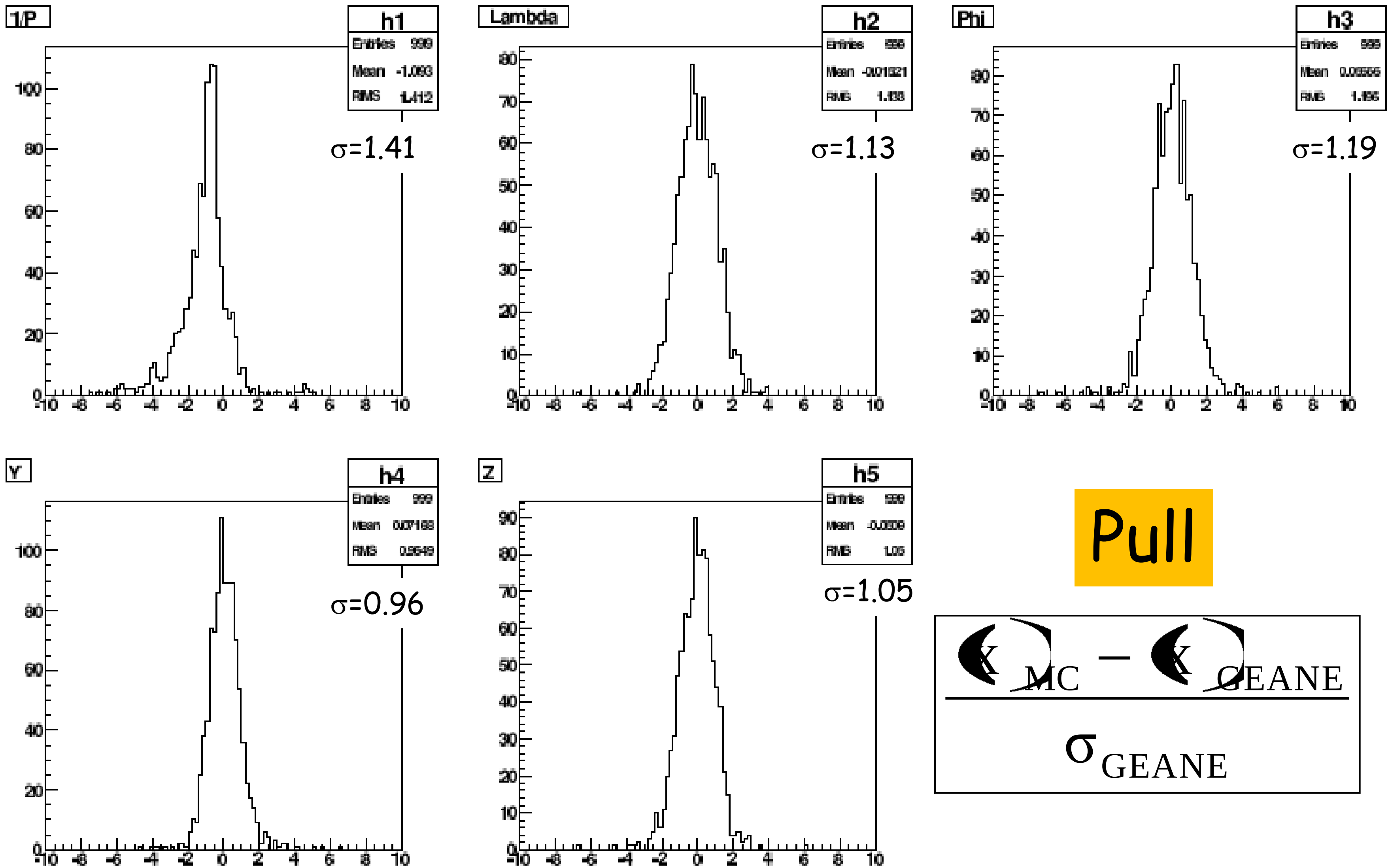
absorber	energy (GeV)	Heitler equation μ	Heitler equation σ	GEANT3 μ	GEANT3 σ	GEANT4 μ	GEANT4 σ
10 cm Ar	0.5	0.4995	0.0097	0.4995	0.0097	0.4995	0.0105
10 cm Ar	1.0	0.9991	0.0194	0.9991	0.0198	0.9991	0.0203
1 cm Al	0.5	0.447	0.098	0.444	0.100	0.444	0.098
1 cm Al	1.0	0.894	0.195	0.891	0.203	0.891	0.201
1 cm Al	10	9.01	1.95	8.96	2.04	8.95	2.06

Table 2: comparison between the mean energy μ and standard deviation σ (MeV) from the the GEANT3 and GEANT4 simulated distributions relative to 10^5 electrons and from the Heitler formula after passing some absorbers.

Bremsstrahlung

$$\begin{aligned}\sigma[1/E] &= 0.5\,[1/E_2, 1/E_1]\ , \qquad \text{where} \\ E_2 &= \mathrm{Min}(E_0, \langle E\rangle + \sigma[E])\ , \\ E_1 &= \begin{cases} \langle E\rangle - \sigma[E] & \text{if} \quad E_2 = \langle E\rangle + \sigma[E] \\ E_0 - 2\sigma[E] & \text{if} \quad E_2 = E_0 \end{cases} \quad .\end{aligned}$$

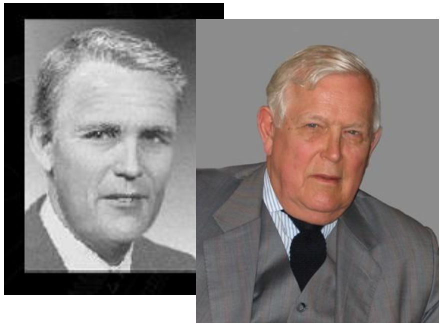




$$\frac{(X)_{MC} - (X)_{GEANE}}{\sigma_{GEANE}}$$

Figure 11: Pull distributions of the 5 track parameters in the case of 2 GeV muons that have passed through the whole detector, just before the PANDA

A Bayesian technique:The KALMAN filter



Consider the well-known weighted mean:

$$\chi^2(\mu) = \frac{(x_1-\mu)^2}{\sigma_1^2} + \frac{(x_2-\mu)^2}{\sigma_2^2}, \qquad \frac{\partial \chi^2(\mu)}{\partial \mu} = 0 \Rightarrow \mu = \frac{\dfrac{x_1}{\sigma_1^2} + \dfrac{x_2}{\sigma_2^2}}{\dfrac{1}{\sigma_1^2} + \dfrac{1}{\sigma_2^2}}$$

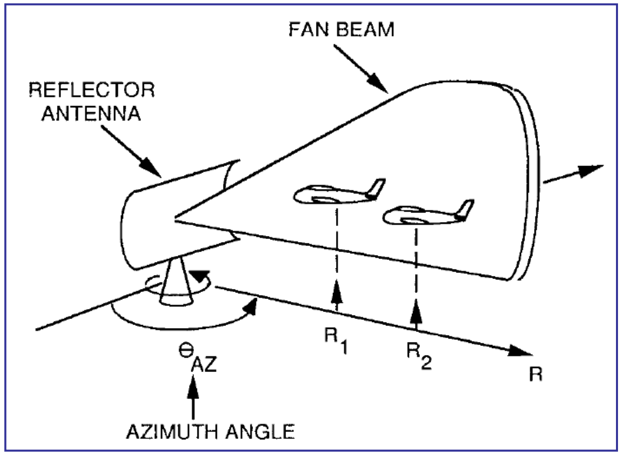
A simple algebraic manipulation gives the recursive form:

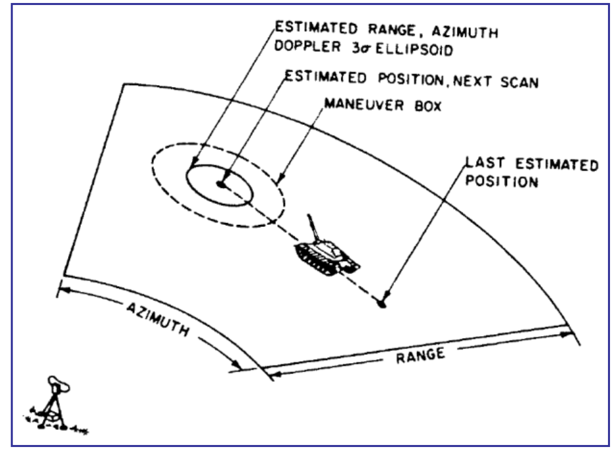
$$\mu = \frac{\dfrac{x_1}{\sigma_1^2} + \dfrac{x_2}{\sigma_2^2}}{\dfrac{1}{\sigma_1^2} + \dfrac{1}{\sigma_2^2}} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\left(\frac{x_1}{\sigma_1^2} + \frac{x_2}{\sigma_2^2}\right) = \frac{x_1\sigma_2^2 + x_2\sigma_1^2}{\sigma_1^2+\sigma_2^2} = x_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}(x_2 - x_1)$$

Kalman= the measurement is weighted with a model prediction (track following)

prediction

Example: Radar Applications





In a radar application, where one is interested in following a target, information about the location, speed, and acceleration of the target is measured at different moments in time with corruption by noise.

State vector

$$r = \{ x, y, z, v_x, v_y, v_z \}$$

position (x, y, z) — velocity (v_x, v_y, v_z)

error of x

Covariance matrix

$$C = \begin{Bmatrix} \sigma^2_x & & & & \cdots & \\ & \sigma^2_y & & & & \\ & & \sigma^2_z & & & \\ & & & \sigma^2_{vx} & & \\ & \cdots & & & \sigma^2_{vy} & \\ & & & & & \sigma^2_{vz} \end{Bmatrix}$$

Apollo Flight Journal

December 21, 1968. The Apollo 8 spacecraft has just been sent on its way to the Moon.

003:46:31 Collins: Roger. At your convenience, would you please go P00 and Accept? We're going to update to your W-matrix.

The original idea is very simple

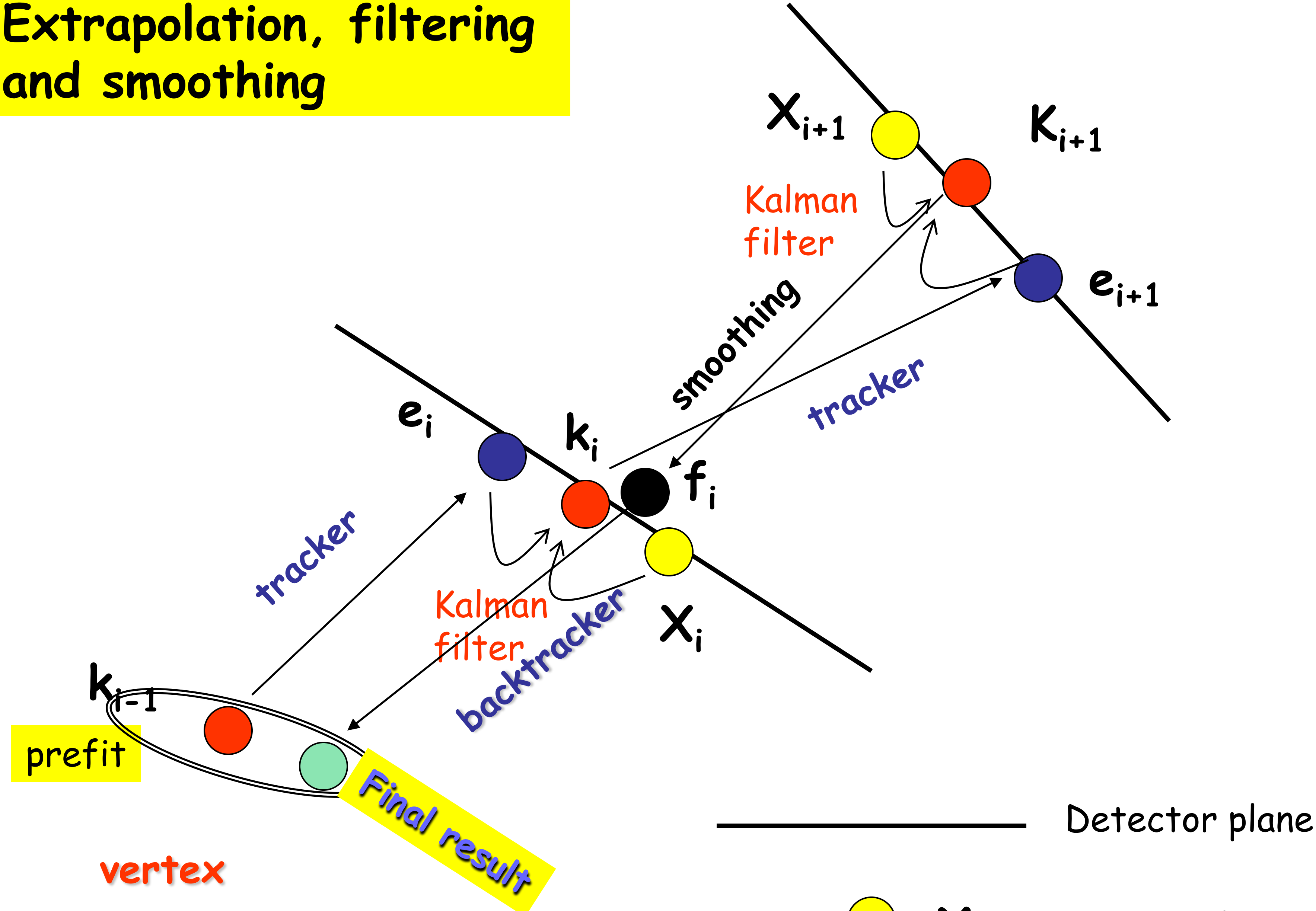
When m is measured at t_2 and $x(t_1,t_2)$ is the **prediction** from t_1 to t_2, the best evaluation of x at t_2 is

This is called the Kalman filter recursive form

$$x(t_2) = m + \frac{\sigma_m^2}{\sigma_m^2 + \sigma_2^2}\left(x(t_1,t_2) - m\right)$$

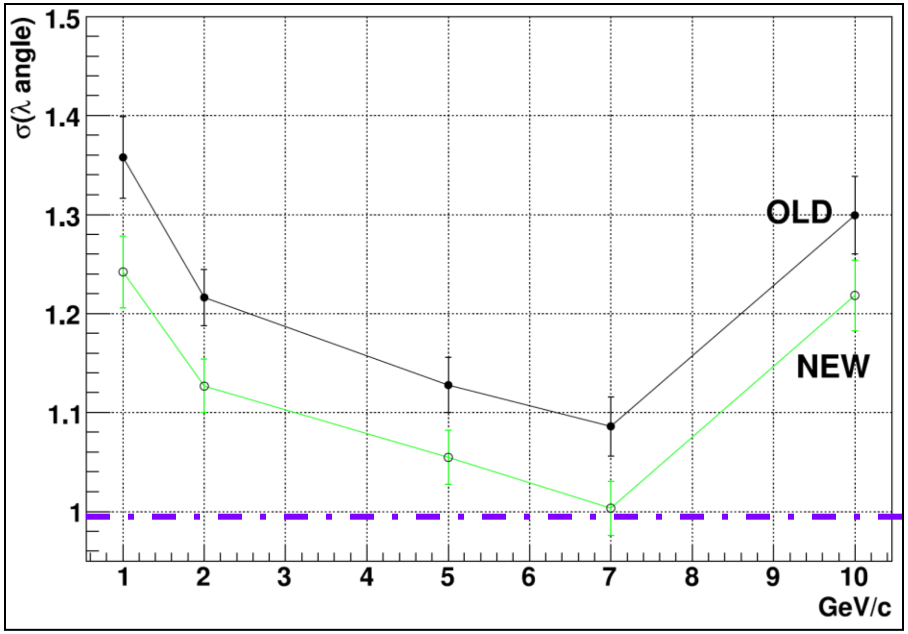
Kalman= the measurement is weighted with a model prediction (track following)

Extrapolation, filtering and smoothing

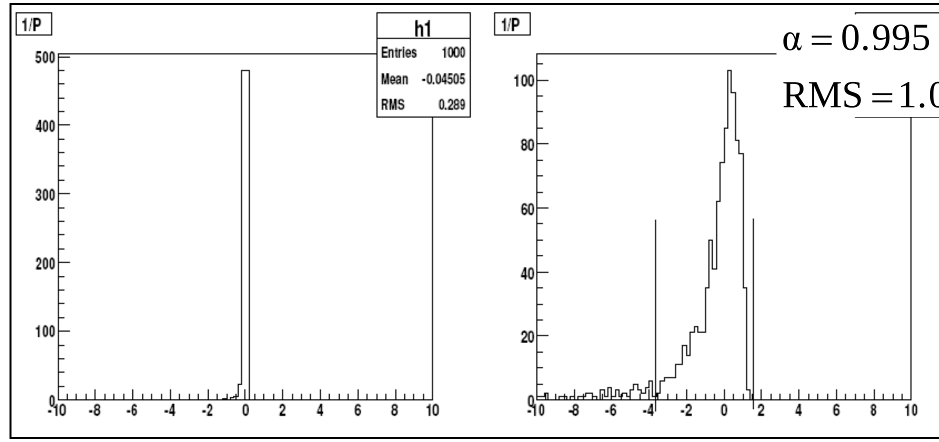


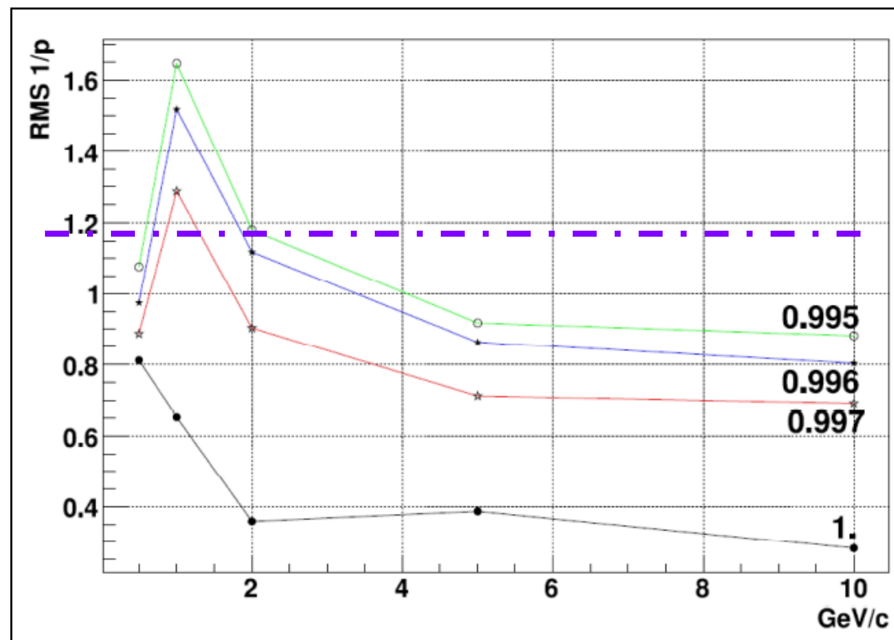
Track propagation: physical effects

Multiple scattering



Energy loss straggling

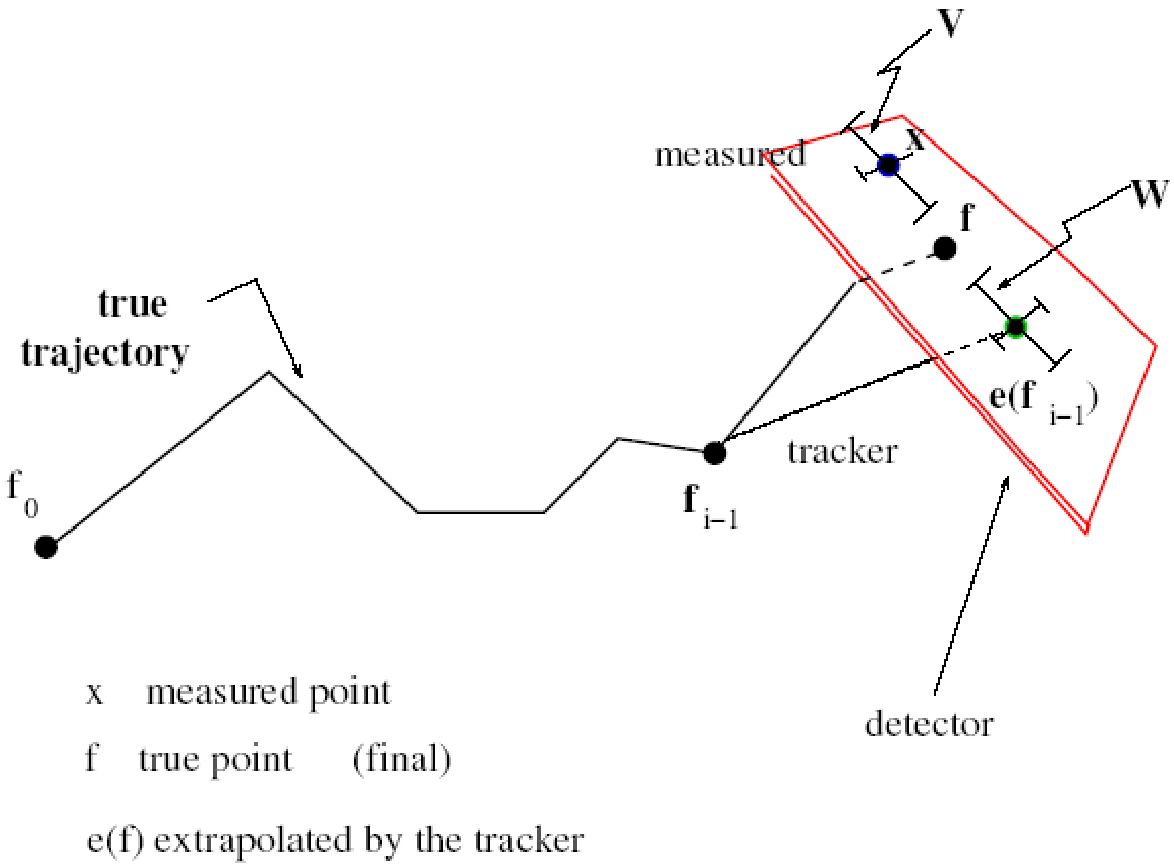




Pull

$$\frac{(X)_{MC} - (X)_{GEANE}}{\sigma_{GEANE}}$$

Tracking with Kalman



The best estimate of the track is given by minimizing w.r.t the f variables:

$$\chi^2(f) = \sum_i [(e_i[f_{i-1}] - f_i) \boldsymbol{W}_{i-1} (e_i[f_{i-1}] - f_i)] + (x_i - f_i) \boldsymbol{V}_i (x_i - f_i) \quad (1)$$

Note the $\boldsymbol{W}$ matrix associated to e_i because the extrapolation start from the true point.

The minimization gives:

$$\frac{\partial \chi^2}{\partial f_i} = \boldsymbol{W}_{i-1,i}(e_i[f_{i-1}] - f_i) + \boldsymbol{V}(x_i - f_i) \qquad (2)$$
$$+\boldsymbol{T}_{i,i+1}\boldsymbol{W}_{i,i+1}(e_{i+1}[f_i] - f_{i+1}) = 0$$

where the last (*extra*) term comes from the extrapolation procedure (tracker).

The best way to solve eq (2) is the Kalman algorithm (Kalman, 1961). It is based on three steps:

V. Innocente and E. Nagy, NIM A324(1993)297
(see their sect. 4.3. Correct their eq. below the (34) one with our eq.(9)).

- **EXTRAPOLATION**: calculation of e_i and $\boldsymbol{W}$. Deterministic step made by the tracker.

$$e_i = \boldsymbol{G}_{i-1,i}[k_{i-1}] \quad (3)$$

$$\boldsymbol{\sigma}^2[e_i] = \boldsymbol{T}_{i-1,i}\,\boldsymbol{\sigma}^2[k_{i-1}]\,\boldsymbol{T}^T_{i-1,i} + \boldsymbol{W}^{-1}_{i-1,i} \quad (4)$$

Square brackets mean function argument

e_i = EXTRAP. extrapolation

k_i = result of the Kalman filter

$\boldsymbol{T}_{i-1,i}$ = EXTRAP. transport matrix

$\boldsymbol{\sigma}(k)^2$ = Kalman error matrix

$\boldsymbol{\sigma}^2[e_i]$ = EXTRAP. error matrix

$\boldsymbol{W}_{i-1,i}$ = EXTRAP. energy loss and multiple scattering weight matrix

$\boldsymbol{W}^{-1}_{i-1,i}$= covariance matrix inverse of W

- **FILTERING**: minimizes the first two terms of eq. (2). It is simply the weighted mean;

$$k_i = \boldsymbol{\sigma}^2[k_i]\left(\boldsymbol{\sigma}^{-2}[e_i]\,e_i + \boldsymbol{V}_i x_i\right) \quad (5)$$

$$\boldsymbol{\sigma}^{-2}[k_i] = \boldsymbol{\sigma}^{-2}[e_i] + \boldsymbol{V}_i \quad (6)$$

x_i = measured points

k_i = Kalman average value

$\boldsymbol{\sigma}(k)^2$ = Kalman error matrix

$\boldsymbol{\sigma}^2(e)$ = EXTRAP. error matrix

$\boldsymbol{V}$ = original weight matrix of the measured points

$$\mu = \frac{\dfrac{x_1}{\sigma_1^2} + \dfrac{x_2}{\sigma_2^2}}{\dfrac{1}{\sigma_1^2} + \dfrac{1}{\sigma_2^2}}$$

- **SMOOTHING**: necessary to minimize a χ^2 in the presence of the extrapolation term (last term in eq. (2)).

$$f_i = k_i + \boldsymbol{A}_i \left(f_{i+1} - e_{i+1}\right) \tag{7}$$

$$\boldsymbol{\sigma}^2[f_i] = \boldsymbol{\sigma}^2[k_i] + \boldsymbol{A}_i \left(\boldsymbol{\sigma}^2[f_{i+1}] - \boldsymbol{\sigma}^2[e_{i+1}]\right) \boldsymbol{A}_i^T \tag{8}$$

$$\boldsymbol{A}_i = \boldsymbol{\sigma}^2[k_i]\, \boldsymbol{T}_{i,i+1}^T\, \boldsymbol{\sigma}^{-2}[e_{i+1}] \tag{9}$$

f_i = final average value
$\boldsymbol{\sigma}(k)^2$ = Kalman error matrix
$\boldsymbol{\sigma}^2(e)$ = EXTRAP. error matrix

Track fitting tools

1. the GEANT3-GEANE old chain:
The mathematics is that of Wittek (EMC Collaboration)
The tracking banks and routines are the same as in MC.
The user gives the starting and ending
planes or volumes and the **tracking is done automatically**.
It works very well (see the CERN Report W5013 GEANE, 1991).

2. "Modern" experiments:
in the software are implemented some tracking classes:
input: x_i, T_i , σ_i, step, medium, magnetic field
output: new x_i, T_i, σ_i
the user has to manage geometry,medium and detector interface

3. A GEANT4-GEANT4E chain there exists in the new GEANT Root framework.
It is used by CMS but is **not included into the official releases**
(see Pedro Arce's talks in the Web)