

- $$\begin{aligned} \frac{d}{dt}\langle x \rangle &= \int dx x \frac{\partial}{\partial t} |\Psi|^2 = - \int dx x \frac{\partial j}{\partial x} = \int dx j \\ &= -\frac{i\hbar}{2m} \int dx \left( \Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right) \\ &= 2 \operatorname{Re} \left[ -\frac{i\hbar}{2m} \int dx \Psi^* \frac{\partial \Psi}{\partial x} \right] \end{aligned}$$

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$$\begin{aligned} 2 \operatorname{Im} \left[ -\frac{i\hbar}{m} \int dx \Psi^* \frac{\partial \Psi}{\partial x} \right] &= -i\hbar \int dx \Psi^* \frac{\partial \Psi}{\partial x} - i\hbar \int dx \Psi \frac{\partial \Psi^*}{\partial x} \\ &= -i\hbar \int dx \frac{\partial}{\partial x} |\Psi|^2 = -i\hbar \left[ |\Psi|^2 \right]_{-\infty}^{+\infty} = 0 \end{aligned}$$

$$\Rightarrow \frac{d}{dt}\langle x \rangle = -\frac{i\hbar}{m} \int dx \Psi^* \frac{\partial \Psi}{\partial x} \equiv \frac{1}{m} \langle p_x \rangle \quad \text{reale}$$

- $$\begin{aligned} \frac{d}{dt}\langle p_x \rangle &= -i\hbar \frac{d}{dt} \int dx \Psi^* \frac{\partial \Psi}{\partial x} \\ &= -i\hbar \int dx \frac{\partial \Psi^*}{\partial t} \frac{\partial \Psi}{\partial x} - i\hbar \int dx \Psi^* \frac{\partial}{\partial x} \frac{\partial \Psi}{\partial t} \\ &= \int dx \left( -\frac{\hbar^2}{2m} \nabla^2 \Psi^* + V \Psi^* \right) \frac{\partial \Psi}{\partial x} - \int dx \Psi^* \frac{\partial}{\partial x} \left( -\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi \right) \\ &= \int dx \Psi^* \left[ V \frac{\partial \Psi}{\partial x} - \frac{\partial}{\partial x} (V \Psi) \right] = - \int dx \Psi^* \frac{\partial V}{\partial x} \Psi \end{aligned}$$

$$\Rightarrow \frac{d}{dt}\langle p_x \rangle = \left\langle -\frac{\partial V(x)}{\partial x} \right\rangle$$