Transverse momentum dependence in parton densities

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Contents

Notations and conventions

The conventions will mainly follow the book of Peskin and Schroeder [78]. We use the metric tensor

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1}$$

with Greek indices running over 0,1,2,3. We define the antisymmetric tensor so that

$$\epsilon^{0123} = +1, \qquad\qquad \epsilon_{0123} = -1. \tag{2}$$

Repeated indices are summed in all cases.

Light-cone vectors

Light-cone vectors will be indicated as

$$a^{\mu} = \left[a^{-},\ a^{+},\ \boldsymbol{a}_T\right]. \tag{3}$$

The dot-product in light-cone components is

$$a \cdot b = a^{+}b^{-} + a^{-}b^{+} - \boldsymbol{a}_T \cdot \boldsymbol{b}_T \tag{4}$$

We introduce the projector on the transverse subspace

$$g_T^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

The light-cone decomposition of a vector can be written in a Lorentz covariant fashion using two light-like vectors n_+ and n_- satisfying $n_\pm^2 = 0$ and $n_+ \cdot n_- = 1$ and promoting $\boldsymbol{a}_T$ to a four-vector $a_T^\mu = [0, 0, \boldsymbol{a}_T]$ so that

$$a^{\mu} = a^{+}n_+^{\mu} + a^{-}n_-^{\mu} + a_T^{\mu}, \tag{6}$$

where

$$
\begin{aligned}
a^{+} &= a \cdot n_{-}, \\
a^{-} &= a \cdot n_{+}, \\
a_T \cdot n_{+} &= a_T \cdot n_{-} = 0.
\end{aligned} \tag{7}
$$

Note that

$$
a_T \cdot b_T = -\boldsymbol{a}_T \cdot \boldsymbol{b}_T \tag{8}
$$

We further have

$$
g_T^{\alpha\beta} = g^{\alpha\beta} - n_+^{\alpha} n_-^{\beta} - n_-^{\alpha} n_+^{\beta}, \tag{9}
$$

and we can define the transverse antisymmetric tensor

$$
\epsilon_T^{\alpha\beta} = \epsilon^{\alpha\beta\rho\sigma} n_{+\rho} n_{-\sigma} \tag{10}
$$

Dirac matrices

Dirac matrices will be often expressed in the chiral or Weyl representation, i.e.

$$
\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\boldsymbol{\sigma}^i \\ \boldsymbol{\sigma}^i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{11}
$$

and we will make use of the Dirac structure

$$
\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{2} \left[\gamma^{\mu}, \gamma^{\nu} \right]. \tag{12}
$$

1

Semi-inclusive DIS at leading twist

This chapter is based essentially on Refs. 18 and 23.

We consider the process

$$\ell(l) + N(P) \to \ell(l') + h(P_h) + X, \tag{1.1}$$

where ℓ denotes the beam lepton, N the nucleon target, and h the produced hadron, and where four-momenta are given in parentheses. We neglect the lepton mass. We denote by M and M_h respective masses of the nucleon and of the hadron h. As usual we define $q = l - l'$ and $Q^2 = -q^2$ and introduce the variables

$$x_B = \frac{Q^2}{2\,P \cdot q}, \qquad y = \frac{P \cdot q}{P \cdot l}, \qquad z_h = \frac{P \cdot P_h}{P \cdot q}. \tag{1.2}$$

In this chapter, we will systematically neglect all correction of order M/Q, unless otherwise specified.

The spin vector of the target is denoted by S. Our definition of the azimuthal angles ϕ_h and ϕ_S of the outgoing hadron and the target spin is shown in Fig. 1.1 and consistent with the Trento conventions [22]. The helicity of the lepton beam is denoted by λ_e. We consider the case where the detected hadron h has spin zero or where its polarization is not measured. $P_{h\perp}$ and $S_\perp$ are the transverse parts P_h and S with respect to the photon momentum. $S_\parallel$ is the component of S in the negative z-direction in Fig. 1.1, i.e. positive $S_\parallel$ corresponds to the target spin pointing towards the virtual photon. In this chapter, we will use without distinguishing them $S_\perp \leftrightarrow S_T$ and $S_\parallel \leftrightarrow S_L$.

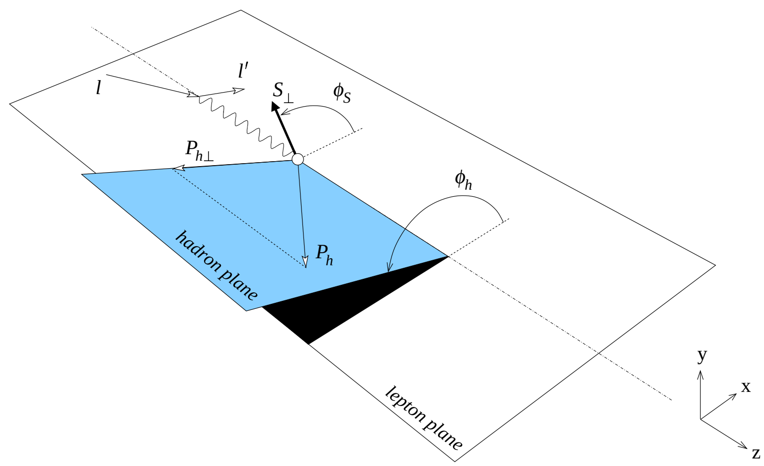


Figure 1.1. Definition of azimuthal angles for semi-inclusive deep inelastic scattering in the target rest frame. $P_{h\perp}$ and $S_\perp$ are the transverse parts of P_h and S with respect to the photon momentum [22].

1.1 Inclusive DIS

1.1.1 Hadronic tensor and structure functions for inclusive DIS

The cross section for polarized electron-nucleon scattering can be written in a general way as the contraction between a leptonic and a hadronic tensor

$$\frac{\mathrm{d}^3\sigma}{\mathrm{d}x_B\,\mathrm{d}y\,\mathrm{d}\phi_S} = \frac{\alpha^2}{2\,s\,x_B\,Q^2}\,L_{\mu\nu}(l,l',\lambda_e)\;2MW^{\mu\nu}(q,P,S), \tag{1.3}$$

where $\alpha = e^2/4\pi$.

Considering the lepton to be longitudinally polarized, in the massless limit the leptonic tensor is given by [76]

$$\begin{aligned} L_{\mu\nu} &= \sum_{\lambda_e'}\Big(\bar{u}(l',\lambda_e')\,\gamma_\mu\,u(l,\lambda_e)\Big)^*\,\Big(\bar{u}(l',\lambda_e')\,\gamma_\nu\,u(l,\lambda_e)\Big) \\ &= -Q^2 g_{\mu\nu} + 2\left(l_\mu l'_\nu + l'_\mu l_\nu\right) + 2\mathrm{i}\,\lambda_e\,\epsilon_{\mu\nu\rho\sigma}\,l^\rho l'^\sigma. \end{aligned} \tag{1.4}$$

The leptonic tensor contains all the information on the leptonic probe, which can be described by means of perturbative QED, while the information on the hadronic target is contained in the hadronic tensor

$$2MW^{\mu\nu}(q,P,S) = \frac{1}{2\pi}\sum_{\mathcal{X}}\int\frac{\mathrm{d}^3\boldsymbol{P}_{\mathcal{X}}}{(2\pi)^3\,2P^0_{\mathcal{X}}}\,(2\pi)^4\,\delta^{(4)}\!\Big(q+P-P_{\mathcal{X}}\Big)\,H^{\mu\nu}(P,S,P_{\mathcal{X}}), \tag{1.5}$$

$$H^{\mu\nu}(P,S,P_{\mathcal{X}}) = \langle P,S\,|\,J^\mu(0)\,|\mathcal{X}\rangle\langle\mathcal{X}|\,J^\nu(0)\,|P,S\rangle. \tag{1.6}$$

The state $\mathcal{X}$ symbolizes any final state, with total momentum $P_{\mathcal{X}}$. It is integrated over since in inclusive processes the final state goes undetected.

In general, the structure of the hadronic tensor can be parametrized in terms of the *structure functions*. In the case of inclusive DIS we can introduce four structure functions, leading to the expression

$$\begin{aligned}\frac{d\sigma}{dx_B\,dy\,d\phi_S} = \frac{2\alpha^2}{x_B y Q^2}\,\frac{y^2}{2\,(1-\varepsilon)}\Big\{F_T + \varepsilon F_L + S_{\parallel}\lambda_e\,\sqrt{1-\varepsilon^2}\,2x_B\,(g_1 - \gamma^2 g_2)\\ - |\boldsymbol{S}_\perp|\lambda_e\,\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_S\,2x_B\gamma\,(g_1 + g_2)\Big\},\end{aligned} \tag{1.7}$$

where the structure functions on the r.h.s. depend on x_B and Q^2 (i.e. $P \cdot Q$ and q^2). We also introduced the ratio ε of longitudinal and transverse photon flux in

$$\varepsilon = \frac{1-y}{1-y+\frac{1}{2}y^2}, \qquad \text{and} \qquad \gamma = \frac{2Mx_B}{Q} \tag{1.8}$$

Ex. 1.1

Eq. (1.7) does not yet look as the standard results in the literature, e.g., Eq. (2.7) in [71]. Check the correspondence by expressing the results with respect to the lepton beam direction.

1.1.2 A convenient frame

For the treatment of inclusive DIS, it is convenient to choose a frame where the proton and photon momenta have no transverse components. In terms of light-cone vectors, it means

$$P^\mu = P^+ n_+^\mu + \frac{M^2}{2P^+}\, n_-^\mu\,, \tag{1.9}$$

$$q^\mu = -x_B P^+ n_+^\mu + \frac{Q^2}{2x_B P^+}\, n_-^\mu\,. \tag{1.10}$$

The spin vector of the target can then be decomposed as

$$S^\mu = S_L\,\frac{(P\cdot n_-)\,n_+^\mu - (P\cdot n_+)\,n_-^\mu}{M} + S_T^\mu\,. \tag{1.11}$$

It is particularly convenient to work in a reference frame where

$$xP^+ = Q/\sqrt{2}. \tag{1.12}$$

Ex. 1.2

Derive the following expressions of the involved momenta in the frame we are using

$$P^\mu = \left[\frac{x_B M^2}{Q\sqrt{2}},\ \frac{Q}{x_B\sqrt{2}},\ \mathbf{0}\right] \tag{1.13a}$$

$$q^\mu = \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \tag{1.13b}$$

$$l^\mu = \left[\frac{Q}{y\sqrt{2}},\ \frac{(1-y)Q}{y\sqrt{2}},\ \frac{Q\sqrt{1-y}}{y},\ 0\right], \tag{1.13c}$$

$$l'^\mu = \left[\frac{(1-y)Q}{y\sqrt{2}},\ \frac{Q}{y\sqrt{2}},\ \frac{Q\sqrt{1-y}}{y},\ 0\right]. \tag{1.13d}$$

1.1.3 Hadronic tensor in the parton model

The phenomenology of DIS taught us that at sufficiently high Q^2 we can assume that the scattering of the electron takes place off a quark of mass m inside the nucleon. The final state $\mathcal{X}$ can be split in a quark with momentum k plus a state X with momentum P_X. Considering the electron-quark interaction at tree level only, the hadronic tensor can be written as

$$\begin{aligned} 2MW^{\mu\nu}(q,P,S) = {} & \frac{1}{2\pi}\sum_q e_q^2 \sum_X \int \frac{\mathrm{d}^3\boldsymbol{P}_X}{(2\pi)^3\,2P_X^0} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\,2k^0}\,(2\pi)^4\,\delta^{(4)}\big(P+q-k-P_X\big) \\ & \times \langle P,S\,|\,\bar{\psi}_i(0)\,|X\rangle\langle X|\,\psi_j(0)\,|P,S\,\rangle\,\gamma^\mu_{ik}\,(\not{k}+m)_{kl}\,\gamma^\nu_{lj}, \end{aligned} \tag{1.14}$$

where k is the momentum of the struck quark, the index q denotes the quark flavor and e_q is the fractional charge of the quark. Note that, for simplicity, we omitted the flavor indices on the quark fields. The integration over the phase space of the final-state quark can be replaced by a four-dimensional integral with an on-shell condition,

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{2k^0} \longrightarrow \int \mathrm{d}^4k\,\delta\big(k^2-m^2\big)\,\theta\big(k^0-m\big), \tag{1.15}$$

so that the hadronic tensor can be rewritten as

$$\begin{aligned} 2MW^{\mu\nu}(q,P,S) = {} & \sum_q e_q^2 \sum_X \int \frac{\mathrm{d}^3\boldsymbol{P}_X}{(2\pi)^3\,2P_X^0} \int \mathrm{d}^4k\,\delta\big(k^2-m^2\big)\,\theta\big(k^0-m\big) \\ & \times \delta^{(4)}\big(P+q-k-P_X\big)\langle P,S\,|\,\bar{\psi}_i(0)\,|X\rangle\langle X|\,\psi_j(0)\,|P,S\,\rangle\,\gamma^\mu_{ik}\,(\not{k}+m)_{kl}\,\gamma^\nu_{lj}. \end{aligned} \tag{1.16}$$

Next, we Fourier transform the Dirac delta function according to

$$\delta^{(4)}\big(P+q-k-P_X\big) \longrightarrow \int \frac{\mathrm{d}^4\xi}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}\,(P+q-k-P_X)\cdot\xi} \tag{1.17}$$

and we introduce the momentum $p = k - q$ to obtain

$$\begin{aligned} 2MW^{\mu\nu}(q,P,S) = \sum_q e_q^2 \sum_X \int \frac{d^3\boldsymbol{P}_X}{(2\pi)^3\, 2P_X^0} \int d^4p\, \delta\Big((p+q)^2 - m^2\Big)\, \theta\Big(p^0 + q^0 - m\Big) \\ \times \int \frac{d^4\xi}{(2\pi)^4}\, e^{i\,(P-p-P_X)\cdot\xi}\, \langle P,S\,|\,\bar{\psi}_i(0)\,|X\rangle\langle X|\,\psi_j(0)\,|P,S\,\rangle\, \gamma^\mu_{ik}\,(\not{p} + \not{q} + m)_{kl}\,\gamma^\nu_{lj}. \end{aligned} \tag{1.18}$$

Finally, we use part of the exponential to perform a translation of the field operators and we use completeness to eliminate the unobserved X states, so that

$$\begin{aligned} 2MW^{\mu\nu}(q,P,S) = \sum_q e_q^2 \int d^4p\, \delta\Big((p+q)^2 - m^2\Big)\, \theta\Big(p^0 + q^0 - m\Big) \int \frac{d^4\xi}{(2\pi)^4}\, e^{-ip\cdot\xi} \\ \times \Big(\langle P,S\,|\,\bar{\psi}_i(\xi)\,\psi_j(0)\,|P,S\,\rangle\, \gamma^\mu_{ik}\,(\not{p} + \not{q} + m)_{kl}\,\gamma^\nu_{lj}\Big). \end{aligned} \tag{1.19}$$

The hadronic tensor can be written in a more compact way by introducing the quark-quark correlation function Φ

$$\begin{aligned} 2MW^{\mu\nu}(q,P,S) = \sum_q e_q^2 \int d^4p\, \delta\Big((p+q)^2 - m^2\Big)\, \theta\Big(p^0 + q^0 - m\Big) \\ \times \mathrm{Tr}\left[\Phi(p,P,S)\gamma^\mu\,(\not{p} + \not{q} + m)\,\gamma^\nu\right] \end{aligned} \tag{1.20}$$

where

$$\begin{aligned} \Phi_{ji}(p,P,S) &= \frac{1}{(2\pi)^4} \int d^4\xi\, e^{-ip\cdot\xi} \langle P,S\,|\,\bar{\psi}_i(\xi)\,\psi_j(0)\,|P,S\,\rangle \\ &= \sum_X \int \frac{d^3\boldsymbol{P}_X}{(2\pi)^3\, 2P_X^0}\, \langle P,S\,|\,\bar{\psi}_i(0)\,|X\rangle\langle X|\,\psi_j(0)\,|P,S\,\rangle\, \delta^{(4)}\Big(P - p - P_X\Big), \end{aligned} \tag{1.21}$$

As the quark fields should carry a flavor index that we omitted, also the correlation functions are flavor dependent and they should be indicated more appropriately as Φ^q. A graphical representation of the hadronic tensor at tree level in the parton model is given by the so-called *handbag diagram*, depicted in Fig. 1.2 on the following page.

We parametrize the quark momentum p in the following way

$$p^\mu = \left[\frac{p^2 + |\boldsymbol{p}_T|^2}{2xP^+},\; xP^+,\; \boldsymbol{p}_T\right]. \tag{1.22}$$

In our approach, we assume that neither the virtuality of the quark, p^2, nor its transverse momentum squared, $|\boldsymbol{p}_T|^2$, can be large in comparison with the hard scale Q^2. Under these conditions, the quark momentum is soft with respect to the hadron momentum and its relevant component is xP^+. In Eq. (1.20), neglecting terms which are $1/Q$ suppressed, we can use an approximate expression for the delta function

$$\delta\Big((p+q)^2 - m^2\Big) \approx \delta(p^+ + q^+) \approx P^+\,\delta(x - x_B) \tag{1.23}$$

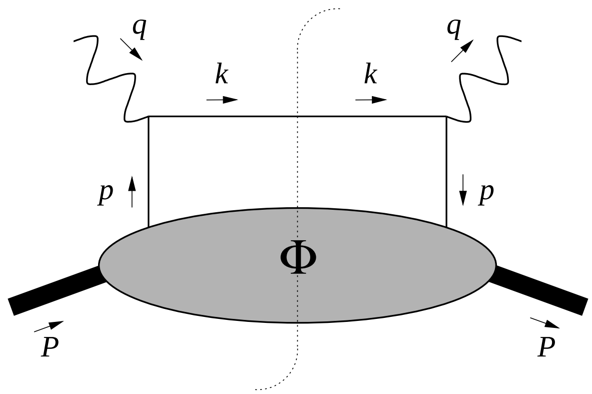


Figure 1.2. The handbag diagram, describing the hadronic tensor for inclusive DIS at Born level.

and replace

$$\mathrm{d}^4 p = \mathrm{d}^2\boldsymbol{p}_T\ \mathrm{d}p^-\ P^+\ \mathrm{d}x \tag{1.24}$$

and obtain

$$\begin{aligned} 2MW^{\mu\nu}(q,P,S) &\approx \sum_q e_q^2 \int \mathrm{d}^2\boldsymbol{p}_T\,\mathrm{d}p^-\,\mathrm{d}x\ \frac{P^+}{2P\cdot q}\ \delta\,(x-x_B)\ \mathrm{Tr}\,[\Phi(p,P,S)\,\gamma^\mu\,(\not p+\not q+m)\,\gamma^\nu] \\ &= \sum_q e_q^2\,\frac{1}{2}\mathrm{Tr}\left[\Phi(x_B,S)\,\gamma^\mu\frac{P^+}{P\cdot q}\,(\not p+\not q+m)\,\gamma^\nu\right] \end{aligned} \tag{1.25}$$

where we introduced the integrated correlation function

$$\begin{aligned} \Phi_{ji}(x,S) &= \int \mathrm{d}^2\boldsymbol{p}_T\,\mathrm{d}p^-\ \Phi_{ji}(p,P,S)\Big|_{p^+=xP^+} \\ &= \int \frac{\mathrm{d}\xi^-}{2\pi}\ \mathrm{e}^{-\mathrm{i}p\cdot\xi}\langle P,S\,|\,\bar\psi_i(\xi)\ \psi_j(0)\,|P,S\rangle\Big|_{\xi^+=\boldsymbol{\xi}_T=0}. \end{aligned} \tag{1.26}$$

Finally, from the outgoing quark momentum, $p+q$, we can select only the minus component and obtain the final form for the hadronic tensor at leading twist

$$2MW^{\mu\nu}(q,P,S) \approx \sum_q e_q^2\,\frac{1}{2}\mathrm{Tr}\,[\Phi(x_B,S)\,\gamma^\mu\gamma^+\gamma^\nu]\,. \tag{1.27}$$

A few words to justify the last approximation are in order. The dominance of the minus component is most easily seen in the infinite momentum frame, where $p^- + q^-$ is of the order of Q, while $p^+ + q^+ = 0$, and $\boldsymbol{p}_T$ and m are of the order of 1. However, if we perform a $1/Q$ expansion of the full expression, including the correlation function Φ [starting from Eq. (2.20)], we would be able to check that in any collinear frame the dominant terms arise only from the combination of plus component in the correlation function and minus components in the outgoing quark momentum.

1.2 Semi-inclusive DIS

1.2.1 Hadronic tensor and structure functions for semi-inclusive DIS

The cross section for one-particle inclusive electron-nucleon scattering can be written as

$$\frac{2E_h\, \mathrm{d}^6\sigma}{\mathrm{d}^3\boldsymbol{P}_h\, \mathrm{d}x_B\, \mathrm{d}y\, \mathrm{d}\phi_S} = \frac{\alpha^2}{2sx_BQ^2}\, L_{\mu\nu}(l,l',\lambda_e)\, 2MW^{\mu\nu}(q,P,S,P_h), \tag{1.28}$$

or equivalently as

$$\frac{\mathrm{d}^6\sigma}{\mathrm{d}x_B\, \mathrm{d}y\, \mathrm{d}z_h\, \mathrm{d}\phi_S\, \mathrm{d}^2\boldsymbol{P}_{h\perp}} = \frac{\alpha^2}{4z_h s x_B Q^2}\, L_{\mu\nu}(l,l',\lambda_e)\; 2MW^{\mu\nu}(q,P,S,P_h). \tag{1.29}$$

To obtain the previous formula, we made use of the relation $\mathrm{d}^3P_h/2E_h \approx \mathrm{d}z_h\, \mathrm{d}^2\boldsymbol{P}_{h\perp}/2z_h$.

The hadronic tensor for one-particle inclusive scattering is defined as

$$2MW^{\mu\nu}(q,P,S,P_h) = \frac{1}{(2\pi)^4}\sum_{X'}\int \frac{\mathrm{d}^3\boldsymbol{P}_{X'}}{2P^0_{X'}}\, 2\pi\, \delta^{(4)}\Big(q+P-P_{X'}-P_h\Big)\, H^{\mu\nu}(P,S,P_{X'},P_h), \tag{1.30}$$

$$H^{\mu\nu}(P,S,P_{X'},P_h) = \langle P,S\,|\, J^\mu(0)\,|P_h,X'\rangle\langle P_h,X'|\, J^\nu(0)\,|P,S\,\rangle. \tag{1.31}$$

In semi-inclusive DIS, the hadronic tensor can be parametrized in terms of 18 structure functions [59], see e.g. Refs. [50, 70]

$$\begin{aligned}
\frac{d\sigma}{dx_B\, dy\, d\phi_S\, dz\, d\phi_h\, dP_{h\perp}^2} = \frac{\alpha^2}{x_B y\, Q^2}\,\frac{y^2}{2\,(1-\varepsilon)}\Bigg\{ & F_{UU,T} + \varepsilon F_{UU,L} + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\, F_{UU}^{\cos\phi_h} \\
& + \varepsilon\cos(2\phi_h)\, F_{UU}^{\cos 2\phi_h} + \lambda_e\, \sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\, F_{LU}^{\sin\phi_h} \\
& + S_\parallel \Bigg[\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_h\, F_{UL}^{\sin\phi_h} + \varepsilon\sin(2\phi_h)\, F_{UL}^{\sin 2\phi_h}\Bigg] \\
& + S_\parallel\, \lambda_e \Bigg[\sqrt{1-\varepsilon^2}\, F_{LL} + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_h\, F_{LL}^{\cos\phi_h}\Bigg] \\
& + |\boldsymbol{S}_\perp| \Bigg[\sin(\phi_h-\phi_S)\Big(F_{UT,T}^{\sin(\phi_h-\phi_S)} + \varepsilon\, F_{UT,L}^{\sin(\phi_h-\phi_S)}\Big) \\
& \quad + \varepsilon\,\sin(\phi_h+\phi_S)\, F_{UT}^{\sin(\phi_h+\phi_S)} + \varepsilon\,\sin(3\phi_h-\phi_S)\, F_{UT}^{\sin(3\phi_h-\phi_S)} \\
& \quad + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_S\, F_{UT}^{\sin\phi_S} + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin(2\phi_h-\phi_S)\, F_{UT}^{\sin(2\phi_h-\phi_S)}\Bigg] \\
& + |\boldsymbol{S}_\perp|\lambda_e \Bigg[\sqrt{1-\varepsilon^2}\,\cos(\phi_h-\phi_S)\, F_{LT}^{\cos(\phi_h-\phi_S)} + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_S\, F_{LT}^{\cos\phi_S} \\
& \quad + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos(2\phi_h-\phi_S)\, F_{LT}^{\cos(2\phi_h-\phi_S)}\Bigg]\Bigg\}.
\end{aligned} \tag{1.32}$$

The first and second subscript of the above structure functions indicate the respective polarization of beam and target, whereas the third subscript in $F_{UU,T}$ and $F_{UU,L}$ specifies the polarization of the virtual photon. Using the definition (1.8), the depolarization factors can be written in terms of the y variable as

$$\frac{y^2}{2\,(1-\varepsilon)} = \left(1 - y + y^2/2\right) \tag{1.33}$$

$$\frac{y^2}{2\,(1-\varepsilon)}\,\varepsilon = (1-y) \tag{1.34}$$

$$\frac{y^2}{2\,(1-\varepsilon)}\,\sqrt{1-\varepsilon^2} = y\,(1-y/2) \tag{1.35}$$

$$\frac{y^2}{2\,(1-\varepsilon)}\,\sqrt{2\,\varepsilon(1+\varepsilon)} = (2-y)\,\sqrt{1-y} \tag{1.36}$$

$$\frac{y^2}{2\,(1-\varepsilon)}\,\sqrt{2\,\varepsilon(1-\varepsilon)} = y\,\sqrt{1-y} \tag{1.37}$$

Integration of Eq. (1.32) over the transverse momentum $\boldsymbol{P}_{h\perp}$ of the outgoing hadron gives the semi-inclusive deep inelastic scattering cross section

$$\begin{aligned}\frac{d\sigma}{dx_B\,dy\,d\phi_S\,dz} = \frac{2\alpha^2}{x_B y Q^2}\,\frac{y^2}{2\,(1-\varepsilon)}\Big\{F_{UU,T} + \varepsilon F_{UU,L} + S_{\parallel}\lambda_e\,\sqrt{1-\varepsilon^2}\,F_{LL} \\ + |\boldsymbol{S}_\perp|\,\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_S\,F_{UT}^{\sin\phi_S} + |\boldsymbol{S}_\perp|\lambda_e\,\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_S\,F_{LT}^{\cos\phi_S}\Big\},\end{aligned} \tag{1.38}$$

where now the structure functions on the r.h.s. are integrated versions of the previous ones, i.e.

$$F_{UU,T}(x_B, z_h, Q^2) = \int \mathrm{d}^2\boldsymbol{P}_{h\perp}\,F_{UU,T}(x_B, z_h, P_{h\perp}^2, Q^2) \tag{1.39}$$

and similarly for the other functions.

Finally, the connection the result for totally inclusive DIS can be obtained by

$$\frac{d\sigma(\ell p \to \ell X)}{dx_B\,dy\,d\phi_S} = \sum_h \int \mathrm{d}z_h\,z_h\,\frac{d\sigma(\ell p \to \ell h X)}{dz\,dx_B\,dy\,d\phi_S} \tag{1.40}$$

where we have summed over all hadrons in the final state. This leads to the result already give in Eq. (1.7), once we identify

$$\sum_h \int \mathrm{d}z_h\,z_h\,F_{UU,T}(x_B, z_h, Q^2) = 2x_B F_1(x_B, Q^2) = F_T(x_B, Q^2), \tag{1.41}$$

$$\sum_h \int \mathrm{d}z_h\,z_h\,F_{UU,L}(x_B, z_h, Q^2) = (1+\gamma^2)F_2(x_B, Q^2) - 2x_B F_1(x_B, Q^2) = F_L(x_B, Q^2), \tag{1.42}$$

$$\sum_h \int \mathrm{d}z_h\,z_h\,F_{LL}(x_B, z_h, Q^2) = 2x_B\,(g_1(x_B, Q^2) - \gamma^2 g_2(x_B, Q^2)), \tag{1.43}$$

$$\sum_h \int \mathrm{d}z_h\,z_h\,F_{LT}^{\cos\phi_S}(x_B, z_h, Q^2) = -2x_B\gamma\,(g_1(x_B, Q^2) + g_2(x_B, Q^2)) \tag{1.44}$$

in terms of the conventional deep inelastic structure functions. Finally, time-reversal invariance requires (see, e.g., Ref. [50])

$$\sum_h \int \mathrm{d}z_h \, z_h \, F_{UT}^{\sin\phi_S}(x_B, z_h, Q^2) = 0. \tag{1.45}$$

1.2.2 Another convenient frame

The choice of a convenient frame to deal with semi-inclusive DIS is less straightforward than for inclusive DIS, due to the presence of P_h. We have two choices:

- FRAME 1: Keep the photon and proton to be collinear, give a transverse component to P_h. This means to keep the parametrization of the vectors as given in Eq. (1.13) and simply adding

$$P_h^\mu = \left[\frac{z_h Q}{\sqrt{2}}, \; \frac{M_h^2 + |\boldsymbol{P}_{h\perp}|^2}{z_h Q \sqrt{2}}, \; \boldsymbol{P}_{h\perp}\right] \tag{1.46}$$

- FRAME 2: Keep the proton and outgoing hadron to be collinear, give a transverse component to q. In terms of light-cone vectors this means choosing

$$P^\mu = P^+ n_+^\mu + \frac{M^2}{2P^+} n_-^\mu \,, \tag{1.47}$$

$$P_h^\mu = P_h^- n_-^\mu + \frac{M_h^2}{2P_h^-} n_+^\mu \,. \tag{1.48}$$

  In this frame, the photon momentum has a transverse component. If we further fix

$$x P^+ = P_h^- / z = Q/\sqrt{2} \tag{1.49}$$

  we can explicitly write the vectors involved as follows

$$P^\mu = \left[\frac{x_B M^2}{Q\sqrt{2}}, \; \frac{Q}{x_B \sqrt{2}}, \; \mathbf{0}\right] \tag{1.50a}$$

$$P_h^\mu = \left[\frac{z_h Q}{\sqrt{2}}, \; \frac{M_h^2}{z_h Q \sqrt{2}}, \; \mathbf{0}\right] \tag{1.50b}$$

$$q^\mu = \left[\frac{Q}{\sqrt{2}}, \; -\frac{\left(Q^2 - |\boldsymbol{q}_T|^2\right)}{Q\sqrt{2}}, \; \boldsymbol{q}_T\right] \approx \left[\frac{Q}{\sqrt{2}}, \; -\frac{Q}{\sqrt{2}}, \; \boldsymbol{q}_T\right] \tag{1.50c}$$

The first choice seems to be the most simple one, but in reality from the theoretical point of view it is better to stick to the second option, in order to preserve a symmetry between P and P_h.

In any case, it turns out that if we neglect subleading twist corrections, all vectors in the two frames are approximately the same, the only difference is the presence of $\boldsymbol{P}_\perp$ in FRAME 1 and the presence of $\boldsymbol{q}_T$ in FRAME 2, and the two are simply connected by

$$\boldsymbol{q}_T = -z \boldsymbol{P}_{h\perp}. \tag{1.51}$$

Therefore, in this chapter we are not going to care very much about distinguishing the two frames, and every time we have $\boldsymbol{q}_T$ we can replace it with $-z\boldsymbol{P}_{h\perp}$ or vice-versa, at our convenience.

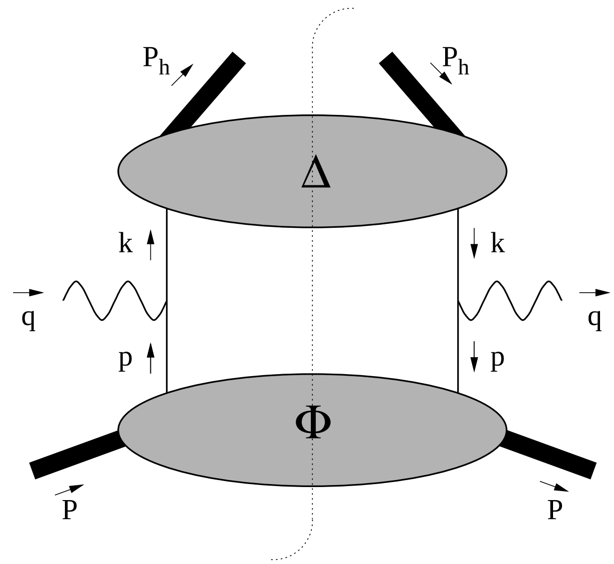


Figure 1.3. The bull diagram, describing the hadronic tensor at tree level.

1.2.3 Hadronic tensor in the parton model for semi-inclusive DIS

In the spirit of the parton model, the virtual photon strikes a quark inside the nucleon. In the case of current fragmentation, the tagged final state hadron comes from the fragmentation of the struck quark. The scattering process can then be factorized in two soft hadronic parts connected by a hard scattering part, as shown in Fig. 1.3.

Considering only the Born-level contribution to the hard scattering, the hadronic tensor can be written as

$$2MW^{\mu\nu}(q,P,S,P_h) = \sum_q e_q^2 \int \mathrm{d}^4p\,\mathrm{d}^4k\;\delta^{(4)}(p+q-k)\;\mathrm{Tr}\left(\Phi(p,P,S)\,\gamma^\mu\,\Delta(k,P_h)\,\gamma^\nu\right), \tag{1.52}$$

where Φ and Δ are so-called quark-quark correlation functions and are defined as

$$\begin{aligned} \Phi_{ji}(p,P,S) &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4\xi\;\mathrm{e}^{\mathrm{i}p\cdot\xi}\langle P,S|\,\bar\psi_i(0)\,\psi_j(\xi)\,|P,S\rangle \\ &= \sum_X \int \frac{\mathrm{d}^3\boldsymbol{P}_X}{(2\pi)^3\,2P_X^0}\;\langle P,S|\,\bar\psi_i(0)\,|X\rangle\langle X|\,\psi_j(0)\,|P,S\rangle\;\delta^{(4)}\big(P-p-P_X\big), \end{aligned} \tag{1.53}$$

$$\begin{aligned} \Delta_{kl}(k,P_h) &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4\xi\;\mathrm{e}^{\mathrm{i}k\cdot\xi}\;\langle 0|\,\psi_k(\xi)\,|P_h\rangle\langle P_h|\,\bar\psi_l(0)\,|0\rangle \\ &= \sum_Y \int \frac{\mathrm{d}^3\boldsymbol{P}_Y}{(2\pi)^3\,2P_Y^0}\;\langle 0|\,\psi_k(0)\,|P_h,Y\rangle\langle P_h,Y|\,\bar\psi_l(0)\,|0\rangle\;\delta^{(4)}\big(k-P_h-P_Y\big). \end{aligned} \tag{1.54}$$

We need to introduce a parametrization for the vectors

$$p^\mu = \left[\frac{p^2+|\boldsymbol{p}_T|^2}{2xP^+},\; xP^+,\; \boldsymbol{p}_T\right], \tag{1.55a}$$

$$k^\mu = \left[\frac{P_h^-}{z},\ \frac{z\left(k^2+|\boldsymbol{k}_T|^2\right)}{2P_h^-},\ \boldsymbol{k}_T\right]. \tag{1.55b}$$

Neglecting terms which are $1/Q$ suppressed, we can write

$$\begin{aligned}\delta^{(4)}\,(p+q-k) &\approx \delta(p^+ + q^+)\,\delta(q^- - k^-)\,\delta^{(2)}\left(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T\right)\\ &\approx \frac{1}{P^+\,P_h^-}\,\delta(x-x_B)\,\delta(1/z-1/z_h)\,\delta^{(2)}\left(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T\right)\end{aligned} \tag{1.56}$$

and replacing

$$\mathrm{d}^4k = \mathrm{d}^2\boldsymbol{k}_T\ \mathrm{d}k^+\,P_h^-\,\frac{\mathrm{d}\,z}{z^2} \tag{1.57}$$

we obtain the compact expression

$$2MW^{\mu\nu}(q,P,S,P_h) = 2z_h\,\mathcal{I}\Big[\mathrm{Tr}(\Phi(x_B,\boldsymbol{p}_T,S)\,\gamma^\mu\,\Delta(z_h,\boldsymbol{k}_T)\,\gamma^\nu)\Big], \tag{1.58}$$

where, as we shall do very often, we used the shorthand notation

$$\mathcal{I}\Big[\cdots\Big] \equiv \int \mathrm{d}^2\boldsymbol{p}_T\,\mathrm{d}^2\boldsymbol{k}_T\,\delta^{(2)}\Big(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T\Big)\Big[\cdots\Big] = \int \mathrm{d}^2\boldsymbol{p}_T\,\mathrm{d}^2\boldsymbol{k}_T\,\delta^{(2)}\Big(\boldsymbol{p}_T-\frac{\boldsymbol{P}_{h\perp}}{z}-\boldsymbol{k}_T\Big)\Big[\cdots\Big], \tag{1.59}$$

and where we introduced the integrated correlation functions

$$\Phi(x,\boldsymbol{p}_T,S) \equiv \int \mathrm{d}p^-\Phi(p,P,S)\Big|_{p^+=xP^+}, \tag{1.60a}$$

$$\Delta(z,\boldsymbol{k}_T) \equiv \frac{1}{2z}\int \mathrm{d}k^+\Delta(k,P_h)\Big|_{k^-=P_h^-/z}. \tag{1.60b}$$

Integrating the cross section over $\boldsymbol{P}_{h\perp}$ we get

$$\frac{\mathrm{d}^4\sigma}{\mathrm{d}x_B\,\mathrm{d}y\,\mathrm{d}z_h\,\mathrm{d}\phi_S} = \frac{\alpha^2}{4z_h s x_B\,Q^2}\,L_{\mu\nu}(l,l',\lambda_e)\,2MW^{\mu\nu}(q,P,S), \tag{1.61}$$

where

$$2MW^{\mu\nu}(q,P,S) = 2z_h\,\mathrm{Tr}(\Phi(x_B,S)\,\gamma^\mu\,\Delta(z_h)\,\gamma^\nu), \tag{1.62a}$$

$$\Phi(x,S) \equiv \int \mathrm{d}p^-\,\mathrm{d}^2\boldsymbol{p}_T\,\Phi(p,P,S)\Big|_{p^+=xP^+}, \tag{1.62b}$$

$$\Delta(z) \equiv \frac{z}{2}\int \mathrm{d}k^+\,\mathrm{d}^2\boldsymbol{k}_T\,\Delta(k,P_h)\Big|_{k^-=P_h^-/z}. \tag{1.62c}$$

2

The correlation functions

2.1 The correlation function Φ for an unpolarized target

To get more insight into the information contained in the correlation function, which is a Dirac matrix, we can decompose it in a general way on a basis of Dirac structures. Each term of the decomposition can be a combination of the Lorentz vectors p and P, the Lorentz pseudovector S (in case of spin-half hadrons) and the Dirac structures

$$\mathbf{1},\ \gamma_5,\ \gamma^\mu,\ \gamma^\mu\gamma_5,\ \mathrm{i}\sigma^{\mu\nu}\gamma_5.$$

The spin vector can only appear linearly in the decomposition (cf. Eq. (2.27)). Moreover, each term of the full expression has to satisfy the conditions of Hermiticity and parity invariance

$$\text{Hermiticity:} \qquad \Phi(p,P,S) = \gamma^0\,\Phi^\dagger(p,P,S)\,\gamma^0, \tag{2.1a}$$
$$\text{parity:} \qquad \Phi(p,P,S) = \gamma^0\,\Phi(\tilde{p},\tilde{P},-\tilde{S})\,\gamma^0 \tag{2.1b}$$

where $\tilde{p}^\nu = \delta^{\nu\mu}p_\mu$ and so forth for the other vectors.

For an unpolarized target, the most general decomposition is

$$\Phi(p,P) = M\,A_1\,\mathbf{1} + A_2\,\not{P} + A_3\,\not{p} + \frac{A_4}{M}\,\sigma_{\mu\nu}P^\mu p^\nu \tag{2.2}$$

where the amplitudes A_i are real scalar functions $A_i = A_i(p\cdot P, p^2)$ with dimension $1/[m]^4$.

If we keep only the leading terms in $1/P^+$ (which means also leading in $1/Q$, i.e. leading twist)

$$\Phi(p,P) \approx P^+\,(A_2 + xA_3)\,\not{n}_+ + P^+\frac{\mathrm{i}}{2M}[\not{n}_+, \not{p}_T]A_4, \tag{2.3}$$

which implies

$$\Phi(x, p_T) \equiv \int \mathrm{d}p^- \Phi(p, P) = \frac{1}{2}\left\{ f_1 \not{n}_+ + \mathrm{i} h_1^\perp \frac{[\not{p}_T, \not{n}_+]}{2M} \right\}. \tag{2.4}$$

Here we introduced the parton distribution functions

$$f_1(x, p_T^2) = 2P^+ \int \mathrm{d}p^- \,(A_2 + xA_3)\,, \qquad h_1^\perp(x, p_T^2) = 2P^+ \int \mathrm{d}p^- \,(-A_4)\,. \tag{2.5}$$

To make clear what variables the functions depend on, it is better to introduce the symbols $\tau = p^2$ and $\sigma = 2p \cdot P$, which are related by [see e.g. Eq. 1.22]

$$\tau = x\sigma - x^2 M^2 - \boldsymbol{p}_T^2, \tag{2.6}$$

we can then rewrite

$$f_1(x, p_T^2) = \int \mathrm{d}\sigma\, \mathrm{d}\tau\, \delta(\tau - x\sigma + x^2 M^2 + \boldsymbol{p}_T^2)\,[A_2(\sigma, \tau) + xA_3(\sigma, \tau)] \tag{2.7}$$

$$h_1^\perp(x, p_T^2) = \int \mathrm{d}\sigma\, \mathrm{d}\tau\, \delta(\tau - x\sigma + x^2 M^2 + \boldsymbol{p}_T^2)\,[-A_4(\sigma, \tau)]. \tag{2.8}$$

The correlation function can be separated in a *T-even* part and a *T-odd* part, according to the definition

$$\Phi^*_{\text{T-even}}(p, P, S) = \mathrm{i}\gamma^1\gamma^3\, \Phi_{\text{T-even}}(\tilde{p}, \tilde{P}, \tilde{S})\, \mathrm{i}\gamma^1\gamma^3, \tag{2.9a}$$

$$\Phi^*_{\text{T-odd}}(p, P, S) = -\mathrm{i}\gamma^1\gamma^3\, \Phi_{\text{T-odd}}(\tilde{p}, \tilde{P}, \tilde{S})\, \mathrm{i}\gamma^1\gamma^3. \tag{2.9b}$$

Note that the distribution functions $h_1^\perp$ is T-odd. At first, this class of functions was supposed to vanish due to time-reversal invariance [81]. The introduction of $h_1^\perp$ was carried out by Boer and Mulders [28]. A proof of the nonexistence of T-odd distribution functions was suggested by Collins in Ref. 46, but recently it has been repudiated by the same author [47] after Brodsky, Hwang and Schmidt [38] explicitly obtained a nonzero T-odd function in the context of a simple model, as we will see in the next chapter.

Ex. 2.1

Check the the term with the amplitude A_4 is T-odd.

The leading-twist part of the correlator Φ can be projected out using the projector

$$\mathcal{P}_+ = \frac{1}{2}\,\gamma^-\gamma^+, \tag{2.10}$$

Before the interaction with the virtual photon, the relevant components of the quark fields are the plus components, $\psi_+ = \mathcal{P}_+\,\psi$. They are usually referred to as the *good components*. It turns out that

$$\left(\mathcal{P}_+\,\Phi\gamma^+\right) = \left\{ f_1 + \mathrm{i} h_1^\perp \frac{\not{p}_T}{M} \right\} \mathcal{P}_+. \tag{2.11}$$

Finally, it is useful to define the matrix $F = \left(\mathcal{P}_+ \, \Phi\gamma^+\right)^T$, i.e. the Dirac transpose of the leading-twist part of the correlation function, and observe that

$$
\begin{aligned}
F(x, \boldsymbol{p}_T)_{ij} &= \int \frac{\mathrm{d}\xi^- \, \mathrm{d}^2\boldsymbol{\xi}_T}{(2\pi)^3 \sqrt{2}} \, \mathrm{e}^{-\mathrm{i}p\cdot\xi} \langle P| \, (\psi_+)^\dagger_i(\xi) \, (\psi_+)_j(0) \, |P\rangle \Big|_{\xi^+=0} \\
&= \frac{1}{\sqrt{2}} \sum_X \int \frac{\mathrm{d}^3\boldsymbol{P}_X}{(2\pi)^3 \, 2P_X^0} \langle X| \, (\psi_+)_i(0) \, |P\rangle^* \langle X| \, (\psi_+)_j(0) \, |P\rangle \\
&\quad \times \delta\Big((1-x)P^+ - P_X^+\Big) \, \delta^{(2)}\Big(\boldsymbol{p}_T - \boldsymbol{P}_{X\,T}\Big).
\end{aligned}
\tag{2.12}
$$

For any Dirac spinor $|a\rangle$, the expectation value $\langle a| \, F \, |a\rangle$ must be positive (it a modulus squared). In mathematical terms, this means that the matrix is *positive semidefinite*, i.e. the determinant of all the principal minors of the matrix has to be positive or zero. This property will prove to be essential in deriving bounds on the components of the correlation function, i.e. the parton distribution functions.

2.1.1 Correlation function in helicity formalism

We will now examine how it is possible to write the correlation function as a matrix in the chirality space of the good quark fields.

The correlation function is a 4×4 Dirac matrix. However, due to the presence of the projector on the good components of the quark fields, the leading-twist part spans only a 2×2 Dirac subspace. This is evident if we express the Dirac structures of Eq. (2.11) in the chiral or Weyl representation. Using this representation, the correlation function reads

$$
\left(\mathcal{P}_+ \, \Phi(x, \boldsymbol{p}_T)\gamma^+\right)_{ji} = \begin{pmatrix} f_1 & 0 & 0 & \mathrm{i}\mathrm{e}^{\mathrm{i}\phi_p} \dfrac{|\boldsymbol{p}_T|}{M} \, h_1^\perp \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi_p} \dfrac{|\boldsymbol{p}_T|}{M} \, h_1^\perp & 0 & 0 & f_1 \end{pmatrix}.
\tag{2.13}
$$

As shown by this explicit form, it seems that the four-dimensional Dirac space can be reduced to a two-dimensional space, retaining only the nonzero part of the correlation function. The relevant part of the Dirac space is the one corresponding to good quark fields. To show this explicitly, we introduce the chiral projectors $\mathcal{P}_{R/L} = (1 \pm \gamma_5)/2$ and define the good chiral-right and good chiral-left quark spinors, i.e. the normalized projections

$$
u_{+R} = \frac{\mathcal{P}_+\mathcal{P}_R \, u}{|\mathcal{P}_+\mathcal{P}_R \, u|}, \qquad\qquad u_{+L} = \frac{\mathcal{P}_+\mathcal{P}_L \, u}{|\mathcal{P}_+\mathcal{P}_L \, u|}.
\tag{2.14}
$$

Then, we can define a new matrix in the chirality space of the good quark fields

$$
\left(\mathcal{P}_+ \, \Phi\gamma^+\right)_{\chi_1'\chi_1} \equiv u^j_{+\chi_1'} \left(\mathcal{P}_+ \, \Phi\gamma^+\right)_{ji} u^i_{+\chi_1}
\tag{2.15}
$$

Any contraction with bad quark fields vanishes. Explicit computation of the matrix elements yields

$$\left(\mathcal{P}_+\,\Phi\gamma^+\right)_{RR} = u^j_{+R}\left(\mathcal{P}_+\,\Phi\gamma^+\right)_{ji}\,u^{\dagger\,i}_{+R} = f_1, \tag{2.16a}$$

$$\left(\mathcal{P}_+\,\Phi\gamma^+\right)_{LL} = u^j_{+L}\left(\mathcal{P}_+\,\Phi\gamma^+\right)_{ji}\,u^{\dagger\,i}_{+L} = f_1, \tag{2.16b}$$

$$\left(\mathcal{P}_+\,\Phi\gamma^+\right)_{RL} = u^j_{+R}\left(\mathcal{P}_+\,\Phi\gamma^+\right)_{ji}\,u^{\dagger\,i}_{+L} = \frac{\mathrm{i}}{M}\,(p_x + \mathrm{i}p_y)\,h_1^\perp, \tag{2.16c}$$

$$\left(\mathcal{P}_+\,\Phi\gamma^+\right)_{LR} = u^j_{+L}\left(\mathcal{P}_+\,\Phi\gamma^+\right)_{ji}\,u^{\dagger\,i}_{+R} = -\frac{\mathrm{i}}{M}\,(p_x - \mathrm{i}p_y)\,h_1^\perp. \tag{2.16d}$$

The correlation matrix in the good quark chirality space is then

$$\left(\mathcal{P}_+\,\Phi\gamma^+\right)_{\chi_1'\chi_1} = \begin{pmatrix} f_1 & \mathrm{i}e^{\mathrm{i}\phi_p}\,\dfrac{|\boldsymbol{p}_T|}{M}\,h_1^\perp \\ -\mathrm{i}e^{-\mathrm{i}\phi_p}\,\dfrac{|\boldsymbol{p}_T|}{M}\,h_1^\perp & f_1 \end{pmatrix}. \tag{2.17}$$

As we could have expected, this result corresponds simply to taking the full Dirac matrix in Weyl representation, Eq. (2.13), and stripping off the zeros. From the matrix representation in the chirality space it is clear why the function $h_1^\perp$ is defined to be *chiral odd*.

The distribution matrix is clearly Hermitean. When an exponential $e^{\mathrm{i}l'\phi_p}$ appears in the matrix, we have to take into account l' units of angular momentum in the final state. The condition of angular momentum conservation then requires $\chi_1' + l' = \chi_1$. The condition parity conservation is

$$F(x,\boldsymbol{p}_T)_{\chi_1\chi_1'} = (-1)^{l'}\,F(x,\boldsymbol{p}_T)_{-\chi_1\,-\chi_1'}\Big|_{l'\to -l'}. \tag{2.18}$$

The fact that the matrix has to be positive definite allows us to derive the positivity bound

$$\frac{|\boldsymbol{p}_T|}{M}\,|h_1^\perp(x,\boldsymbol{p}_T^2)| \le f_1(x,\boldsymbol{p}_T^2). \tag{2.19}$$

2.2 The integrated correlation function Φ for a polarized target

In the presence of target spin, the most general decomposition of the correlation function Φ imposing Hermiticity and parity invariance is [76, 81]

$$\begin{aligned}\Phi(p,P,S) = {}& M A_1\,\mathbf{1} + A_2\,\not{P} + A_3\,\not{p} + \frac{A_4}{M}\,\sigma_{\mu\nu}P^\mu p^\nu + \mathrm{i}A_5\;p\cdot S\;\gamma_5 \\ & + M A_6\,\not{S}\,\gamma_5 + A_7\,\frac{p\cdot S}{M}\,\not{P}\,\gamma_5 + A_8\,\frac{p\cdot S}{M}\,\not{p}\,\gamma_5 + \mathrm{i}A_9\,\sigma_{\mu\nu}\gamma_5 S^\mu P^\nu \\ & + \mathrm{i}A_{10}\,\sigma_{\mu\nu}\gamma_5 S^\mu p^\nu + \mathrm{i}A_{11}\,\frac{p\cdot S}{M^2}\,\sigma_{\mu\nu}\gamma_5 P^\mu p^\nu + A_{12}\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu p^\rho S^\sigma}{M},\end{aligned} \tag{2.20}$$

where the amplitudes A_i real scalar functions $A_i = A_i(p\cdot P, p^2)$ with dimension $1/[m]^4$. The terms containing the amplitudes A_4, A_5 and A_{12} can be classified as T-odd.

It will turn out that the above expansion is incomplete when we consider also non leading-twist terms (see Ch. 5).

The general expression is

$$\Phi(x,S) = \frac{1}{2}\Big\{ f_1 \not{n}_+ + S_L\, g_1 \gamma_5 \not{n}_+ + h_1 \,\frac{[\not{S}_T, \not{n}_+]\gamma_5}{2} \Big\} \tag{2.21}$$

$$\mathcal{P}_+\, \Phi(x,S)\gamma^+ = \big(f_1(x) + S_L\, g_1(x)\,\gamma_5 + h_1(x)\,\gamma_5\,\not{S}_T\big)\mathcal{P}_+, \tag{2.22}$$

where we introduced the integrated parton distribution functions

$$f_1(x) = \int \mathrm{d}^2\boldsymbol{p}_T\, \mathrm{d}p^2\, \mathrm{d}(2p\cdot P)\; \delta\Big(\boldsymbol{p}_T^2 + x^2M^2 + p^2 - 2xp\cdot P\Big)\,[A_2 + xA_3]\,, \tag{2.23a}$$

$$g_1(x) = \int \mathrm{d}^2\boldsymbol{p}_T\, \mathrm{d}p^2\, \mathrm{d}(2p\cdot P)\; \delta\Big(\boldsymbol{p}_T^2 + x^2M^2 + p^2 - 2xp\cdot P\Big)\Big[-A_6 - \Big(\frac{p\cdot P}{M^2} - x\Big)(A_7 + xA_8)\Big], \tag{2.23b}$$

$$h_1(x) = \int \mathrm{d}^2\boldsymbol{p}_T\, \mathrm{d}p^2\, \mathrm{d}(2p\cdot P)\; \delta\Big(\boldsymbol{p}_T^2 + x^2M^2 + p^2 - 2xp\cdot P\Big)\Big[-A_9 - xA_{10} + \frac{\boldsymbol{p}_T^2}{2M^2}A_{11}\Big]. \tag{2.23c}$$

The function f_1 is usually referred to as the unpolarized parton distribution, and it is sometimes denoted also as simply f or q (where q stands for the quark flavor). The function g_1 is the parton helicity distribution and it can be denoted also as Δf or Δq. Finally, the function h_1 is known as the parton transversity distribution; in the literature it is sometimes denoted as δq, $\Delta_T q$ or $\Delta_T f$, although in the original paper of Ralston and Soper [81] it was called h_T.

The individual distribution functions can be isolated by means of the projection

$$\Phi^{[\Gamma]} \equiv \frac{1}{2}\mathrm{Tr}\,(\Phi\Gamma)\,, \tag{2.24}$$

where Γ stands for a specific Dirac structure. In particular, we see that

$$f_1(x) = \Phi^{[\gamma^+]}(x), \tag{2.25a}$$

$$g_1(x) = \Phi^{[\gamma^+\gamma^5]}(x), \tag{2.25b}$$

$$h_1(x) = \Phi^{[\mathrm{i}\sigma^{i+}\gamma^5]}(x). \tag{2.25c}$$

2.2.1 Correlation function in helicity formalism

We will now examine how it is possible to write the correlation function as a matrix in the chirality space of the good quark fields ⊗ the spin space of the hadron. The steps for the chirality space are analogous to the previous case, but the treatment of the target spin is obviously new.

Using the Weyl representation, the correlation function reads

$$\Big(\mathcal{P}_+\, \Phi(x,S)\gamma^+\Big)_{ji} = \begin{pmatrix} f_1(x) + S_L\, g_1(x) & 0 & 0 & (S_x - \mathrm{i}S_y)\, h_1(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (S_x + \mathrm{i}S_y)\, h_1(x) & 0 & 0 & f_1(x) - S_L\, g_1(x) \end{pmatrix}. \tag{2.26}$$

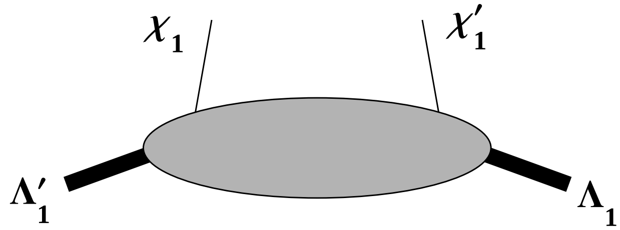


Figure 2.1. Illustration of the position of the indices of the correlation matrix.

The four-dimensional Dirac space can be reduced to a two-dimensional space, retaining only the nonzero part of the correlation function, i.e.

$$\left(\mathcal{P}_+\,\Phi(x,S)\gamma^+\right)_{\chi'_1\chi_1} = \rho(S)_{\Lambda_1\Lambda'_1}\,\left(\mathcal{P}_+\,\Phi(x)\gamma^+\right)^{\Lambda'_1\Lambda_1}_{\chi'_1\chi_1}. \tag{2.27}$$

We will refer to the last term of this relation as the matrix representation of the correlation function or, more simply, as the correlation matrix. Fig. 2.1 shows pictorially the position of the spin indices.

Starting from Eq. (2.22) and using the relation

$$\Psi_U + S_L\Psi_L + S_x\Psi_x + S_y\Psi_y = \rho(S)_{\Lambda_1\Lambda'_1}\begin{pmatrix}\Psi_U+\Psi_L & \Psi_x - \mathrm{i}\Psi_y\\ \Psi_x+\mathrm{i}\Psi_y & \Psi_U-\Psi_L\end{pmatrix}^{\Lambda'_1\Lambda_1} \tag{2.28}$$

we can cast the correlation function in the matrix form

$$\left(\mathcal{P}_+\,\Phi(x)\gamma^+\right)^{\Lambda'_1\Lambda_1} = \begin{pmatrix}\left(f_1(x)+g_1(x)\,\gamma_5\right)\mathcal{P}_+ & h_1(x)\,(\gamma_x-\mathrm{i}\gamma_y)\,\gamma_5\mathcal{P}_+\\ h_1(x)\,(\gamma_x+\mathrm{i}\gamma_y)\,\gamma_5\mathcal{P}_+ & \left(f_1(x)-g_1(x)\,\gamma_5\right)\mathcal{P}_+\end{pmatrix}. \tag{2.29}$$

Finally, by expressing the Dirac structures in Weyl representation and reducing the Dirac space as done before, we obtain the matrix representation of the correlation function

$$\left(\mathcal{P}_+\,\Phi(x)\gamma^+\right)^{\Lambda'_1\Lambda_1}_{\chi'_1\chi_1} = \left(\begin{array}{cc|cc} f_1(x)+g_1(x) & 0 & 0 & 0\\ 0 & f_1(x)-g_1(x) & 2h_1(x) & 0\\ \hline 0 & 2h_1(x) & f_1(x)-g_1(x) & 0\\ 0 & 0 & 0 & f_1(x)+g_1(x)\end{array}\right), \tag{2.30}$$

where the inner blocks are in the hadron helicity space (indices $\Lambda'_1\Lambda_1$), while the outer matrix is in the quark chirality space (indices $\chi'_1\chi_1$).

The form of the correlation matrix can also be established directly from angular momentum conservation (requiring $\Lambda'_1+\chi'_1 = \Lambda_1+\chi_1$) and the conditions of Hermiticity and parity invariance. In matrix language, the condition of parity invariance consists in [64]

$$\left(\mathcal{P}_+\,\Phi(x)\gamma^+\right)^{\Lambda'_1\Lambda_1}_{\chi'_1\chi_1} = \left(\mathcal{P}_+\,\Phi(x)\gamma^+\right)^{-\Lambda'_1\,-\Lambda_1}_{-\chi'_1\,-\chi_1}. \tag{2.31}$$

The most general form of the correlation matrix complying with the previous conditions corresponds to Eq. (2.30).

As mentioned at the end of Sec. 2.7 on page 26, with transposing the quark chirality indices of the correlation matrix we obtain the scattering matrix [19, 20]

$$F(x)^{\Lambda_1'\Lambda_1}_{\chi_1\chi_1'} = \left(\begin{array}{cc|cc} f_1(x)+g_1(x) & 0 & 0 & 2h_1(x) \\ 0 & f_1(x)-g_1(x) & 0 & 0 \\ \hline 0 & 0 & f_1(x)-g_1(x) & 0 \\ 2h_1(x) & 0 & 0 & f_1(x)+g_1(x) \end{array}\right). \tag{2.32}$$

Note that because of the inversion of the quark indices, the lower left block has $\chi_1' = R, \chi_1 = L$ and vice versa for the upper right block. Since this matrix must be positive semidefinite, we can readily obtain the positivity conditions

$$f_1(x) \geq 0, \tag{2.33a}$$

$$|g_1(x)| \leq f_1(x), \tag{2.33b}$$

$$|h_1(x)| \leq \tfrac{1}{2}\big(f_1(x)+g_1(x)\big). \tag{2.33c}$$

The last relation is known as the *Soffer bound* [84].

The probabilistic interpretation of the functions f_1 and g_1 is manifest, since they occupy the diagonal elements of the matrix and they are therefore connected to squares of probability amplitudes

$$f_1(x) = \frac{1}{2}\left(F(x)^{\frac{1}{2}\,\frac{1}{2}}_{R\,R} + F(x)^{\frac{1}{2}\,\frac{1}{2}}_{L\,L}\right) \qquad g_1(x) = \frac{1}{2}\left(F(x)^{\frac{1}{2}\,\frac{1}{2}}_{R\,R} - F(x)^{\frac{1}{2}\,\frac{1}{2}}_{L\,L}\right) \tag{2.34}$$

On the other hand, the transversity distribution is off-diagonal in the helicity basis. This means that it does not describe the square of a probability amplitude, but rather the interference between two different amplitudes

$$h_1(x) = \tfrac{1}{2}F(x)^{\frac{1}{2}\,-\frac{1}{2}}_{R\,L} \tag{2.35}$$

The transversity distribution recovers a probability interpretation if we choose the so-called *transversity basis*, instead of the helicity basis, for both quark and hadron [62, 64]. The transversity basis is formed by the “transverse up” and “transverse down” states. They can be expressed in terms of chirality eigenstates

$$u_{\uparrow} = \tfrac{1}{\sqrt{2}}\left(u_R + u_L\right), \qquad u_{\downarrow} = \tfrac{1}{\sqrt{2}}\left(u_R - u_L\right). \tag{2.36}$$

The same relation holds between the hadron transversity and helicity states.

In the new basis, the scattering matrix takes the form

$$F(x)^{\Lambda_1'\Lambda_1}_{\chi_1\chi_1'} = \left(\begin{array}{cc|cc} f_1(x)+h_1(x) & 0 & 0 & g_1(x)+h_1(x) \\ 0 & f_1(x)-h_1(x) & g_1(x)-h_1(x) & 0 \\ \hline 0 & g_1(x)-h_1(x) & f_1(x)-h_1(x) & 0 \\ g_1(x)+h_1(x) & 0 & 0 & f_1(x)+h_1(x) \end{array}\right), \tag{2.37}$$

and clearly the transversity distribution function can be defined as

$$h_1(x) = \frac{1}{2}\left(F(x)^{\uparrow\uparrow}_{\uparrow\uparrow} - F(x)^{\uparrow\uparrow}_{\downarrow\downarrow}\right). \tag{2.38}$$

2.3 The correlation function Φ for a polarized target

Starting from the general decomposition presented in Eq. (2.20), the leading order part of the transverse-momentum dependent correlation function becomes

$$\Phi(x, p_T) = \frac{1}{2}\Big\{ f_1 \not{n}_+ + f_{1T}^\perp \frac{\epsilon_T^{\rho\sigma} S_{T\rho} p_{T\sigma}}{M} \not{n}_+ + g_{1s}\gamma_5 \not{n}_+ \\ + h_{1T} \frac{[\not{S}_T, \not{n}_+]\gamma_5}{2} + h_{1s}^\perp \frac{[\not{p}_T, \not{n}_+]\gamma_5}{2M} + \mathrm{i}\, h_1^\perp \frac{[\not{p}_T, \not{n}_+]}{2M}\Big\}$$

The distribution functions on the r.h.s. depend on x and p_T^2, except for the functions with subscript s, where we use the shorthand notation [76]

$$g_{1s}(x, p_T) = S_L\, g_{1L}(x, p_T^2) - \frac{S_T \cdot p_T}{M}\, g_{1T}(x, p_T^2) \tag{2.39}$$

and so forth for the other functions. The 2 functions $f_{1T}^\perp$ (Sivers function) and $h_1^\perp$ (Boer-Mulders function) are T-odd [29, 58], i.e. they change sign under "naive time reversal", which is defined as usual time reversal without the interchange of initial and final states. The nomenclature of the distribution functions follows closely that of Ref. [76], sometimes referred to as "Amsterdam notation." We remark that a number of other notations exist for some of the distribution functions, see e.g. Refs. [25,60,81]. In particular, transverse-momentum-dependent functions at leading twist have been widely discussed by Anselmino et al. [9, 14, 15]. The connection between the notation in these papers and the one used here is discussed in App. C of Ref. [14].

The definition of the parton distribution functions in terms of the amplitudes A_i, introduced in Eq. (2.20), can be found elsewhere [65, 72, 85].

Useful relations are

$$\Phi^{[\gamma^+]} = f_1(x, p_T^2) - \frac{\epsilon_T^{\rho\sigma} p_{T\rho} S_{T\sigma}}{M}\, f_{1T}^\perp(x, p_T^2)\,, \tag{2.40}$$

$$\Phi^{[\gamma^+\gamma_5]} = S_L\, g_{1L}(x, p_T^2) - \frac{p_T \cdot S_T}{M}\, g_{1T}(x, p_T^2)\,, \tag{2.41}$$

$$\Phi^{[\mathrm{i}\sigma^{\alpha+}\gamma_5]} = S_T^\alpha\, h_1(x, p_T^2) + S_L \frac{p_T^\alpha}{M}\, h_{1L}^\perp(x, p_T^2) \\ - S_{T\rho} \frac{p_T^\alpha p_T^\rho - \frac{1}{2} p_T^2 g_T^{\alpha\rho}}{M^2}\, h_{1T}^\perp(x, p_T^2) - \frac{\epsilon_T^{\alpha\rho} p_{T\rho}}{M}\, h_1^\perp(x, p_T^2)\,. \tag{2.42}$$

For any transverse-momentum dependent distribution function, it will turn out to be convenient to define the notation

$$f^{(1/2)}(x, \boldsymbol{p}_T^2) \equiv \frac{|\boldsymbol{p}_T|}{2M}\, f(x, \boldsymbol{p}_T^2), \tag{2.43a}$$

$$f^{(n)}(x, \boldsymbol{p}_T^2) \equiv \left(\frac{\boldsymbol{p}_T^2}{2M^2}\right)^n f(x, \boldsymbol{p}_T^2), \tag{2.43b}$$

for n integer. We also need to introduce the function

$$h_1(x, \boldsymbol{p}_T^2) \equiv h_{1T}(x, \boldsymbol{p}_T^2) + h_{1T}^{\perp(1)}(x, \boldsymbol{p}_T^2). \tag{2.44}$$

The connection with the integrated distribution functions defined in Eq. (2.23) is

$$f_1(x) = \int \mathrm{d}^2\boldsymbol{p}_T\; f_1(x, \boldsymbol{p}_T^2), \tag{2.45a}$$

$$g_1(x) = \int \mathrm{d}^2\boldsymbol{p}_T\; g_{1L}(x, \boldsymbol{p}_T^2), \tag{2.45b}$$

$$h_1(x) = \int \mathrm{d}^2\boldsymbol{p}_T\; h_1(x, \boldsymbol{p}_T^2). \tag{2.45c}$$

2.3.1 Correlation function in helicity formalism

As done in the previous section, we can express the transverse momentum dependent correlation function as a matrix in the parton chirality space $\otimes$ target helicity space. To simplify the formulae, it is useful to identify the T-odd functions as imaginary parts of some of the T-even functions, which become then complex scalar functions. The following redefinitions are required:[1]

$$g_{1T} + \mathrm{i}\, f_{1T}^{\perp} \to g_{1T}, \qquad\qquad h_{1L}^{\perp} + \mathrm{i} h_1^{\perp} \to h_{1L}^{\perp}. \tag{2.46}$$

The resulting correlation matrix is [19, 20]

$$F(x, \boldsymbol{p}_T)^{\Lambda_1'\Lambda_1}_{\chi_1\chi_1'} = \left(\begin{array}{cc|cc} f_1 + g_{1L} & \frac{|\boldsymbol{p}_T|}{M}\,\mathrm{e}^{-\mathrm{i}\phi_p}\, g_{1T} & \frac{|\boldsymbol{p}_T|}{M}\,\mathrm{e}^{\mathrm{i}\phi_p}\, h_{1L}^{\perp *} & 2\, h_1 \\ \frac{|\boldsymbol{p}_T|}{M}\,\mathrm{e}^{\mathrm{i}\phi_p}\, g_{1T}^{*} & f_1 - g_{1L} & \frac{|\boldsymbol{p}_T|^2}{M^2}\mathrm{e}^{2\mathrm{i}\phi_p}\, h_{1T}^{\perp} & -\frac{|\boldsymbol{p}_T|}{M}\,\mathrm{e}^{\mathrm{i}\phi_p}\, h_{1L}^{\perp} \\ \hline \frac{|\boldsymbol{p}_T|}{M}\,\mathrm{e}^{-\mathrm{i}\phi_p}\, h_{1L}^{\perp} & \frac{|\boldsymbol{p}_T|^2}{M^2}\mathrm{e}^{-2\mathrm{i}\phi_p}\, h_{1T}^{\perp} & f_1 - g_{1L} & -\frac{|\boldsymbol{p}_T|}{M}\,\mathrm{e}^{-\mathrm{i}\phi_p}\, g_{1T}^{*} \\ 2\, h_1 & -\frac{|\boldsymbol{p}_T|}{M}\,\mathrm{e}^{-\mathrm{i}\phi_p}\, h_{1L}^{\perp *} & -\frac{|\boldsymbol{p}_T|}{M}\,\mathrm{e}^{\mathrm{i}\phi_p}\, g_{1T} & f_1 + g_{1L} \end{array}\right), \tag{2.47}$$

where for sake of brevity we did not explicitly indicate the x and $\boldsymbol{p}_T^2$ dependence of the distribution functions and where ϕ_p is the azimuthal angle of the transverse momentum vector.

The distribution matrix is clearly Hermitean. The condition of angular momentum conservation becomes $\Lambda_1' + \chi_1' + l' = \Lambda_1 + \chi_1$. The condition of parity invariance becomes

$$F(x, \boldsymbol{p}_T)^{\Lambda_1'\Lambda_1}_{\chi_1\chi_1'} = (-1)^{l'}\, F(x, \boldsymbol{p}_T)^{-\Lambda_1'\, -\Lambda_1}_{-\chi_1\, -\chi_1'}\Big|_{l' \to -l'}. \tag{2.48}$$

Bounds to insure positivity of any matrix element can be obtained by looking at the one-dimensional and two-dimensional subspaces and at the eigenvalues of the full matrix.[2] The one-dimensional subspaces give the trivial bounds

$$f_1(x, \boldsymbol{p}_T^2) \geq 0\,, \qquad\qquad \left|g_{1L}(x, \boldsymbol{p}_T^2)\right| \leq f_1(x, \boldsymbol{p}_T^2)\,. \tag{2.49}$$

---

[1]From a rigorous point of view, it would be better to introduce new functions, e.g. $\tilde{g}_{1T}$ and $\tilde{h}_{1L}^{\perp}$, but this would overload the notation.

[2]Cf. Ref. 73 for an earlier discussion on positivity bounds for transverse momentum dependent *structure functions*.

From the two-dimensional subspaces we get

$$|h_1| \leq \frac{1}{2}\left(f_1 + g_{1L}\right) \leq f_1, \tag{2.50a}$$

$$\left|h_{1T}^{\perp(1)}\right| \leq \frac{1}{2}\left(f_1 - g_{1L}\right) \leq f_1, \tag{2.50b}$$

$$\left|g_{1T}^{(1)}\right|^2 \leq \frac{\boldsymbol{p}_T^2}{4M^2}\left(f_1 + g_{1L}\right)\left(f_1 - g_{1L}\right) \leq \left(f_1^{(1/2)}\right)^2, \tag{2.50c}$$

$$\left|h_{1L}^{\perp(1)}\right|^2 \leq \frac{\boldsymbol{p}_T^2}{4M^2}\left(f_1 + g_{1L}\right)\left(f_1 - g_{1L}\right) \leq \left(f_1^{(1/2)}\right)^2, \tag{2.50d}$$

where, once again, we did not explicitly indicate the x and $\boldsymbol{p}_T^2$ dependence to avoid too heavy a notation. Besides the Soffer bound of Eq. (2.50a), now extended to include the transverse momentum dependence, new bounds for the distribution functions are found.

The positivity bounds can be sharpened even further by imposing the positivity of the eigenvalues of the correlation matrix. The complete analysis has been accomplished in Ref. 19 (see also Ref. 20).

2.4 The correlation function Δ

While the distribution correlation function describe the *confinement* of partons inside hadrons, the fragmentation correlation function describes the way a virtual parton “decays” into a hadron plus something else, i.e. $q^* \to hY$. This process is referred to as *hadronization*. It is a clear manifestation of color confinement: the asymptotic physical states detected in experiment must be color neutral, so that quarks have to evolve into hadrons.[3]

The procedure for generating a complete decomposition of the correlation functions closely follows what has been done on the distribution side in Sec. 2.1 on page 13. It is necessary to combine the Lorentz vectors k and P_h with a basis of structures in Dirac space, and impose the condition of Hermiticity and parity invariance. The outcome is

$$\Delta(k, P_h) = M_h\, B_1\, \mathbf{1} + B_2\, \not{P}_h + B_3\, \not{k} + \frac{B_4}{M_h}\, \sigma_{\mu\nu} P_h^\mu k^\nu. \tag{2.51}$$

The amplitudes B_i are dimensionless real scalar functions $B_i = B_i(k \cdot P_h, k^2)$. The T-even and T-odd part of the correlation function Δ can be defined in analogy to Eqs. (2.9). According to those definitions, the last term can be classified as T-odd.

At leading twist, we are interested in the projection $\mathcal{P}_-\, \Delta(z, \boldsymbol{k}_T)\, \gamma^-$. The insertion of the decomposition given in Eq. (2.51) into Eq. (1.60b) and the projection of the leading-twist component leads to

$$\mathcal{P}_-\, \Delta(z, \boldsymbol{k}_T)\, \gamma^- = \left(D_1(z, z^2 \boldsymbol{k}_T^2) + i\, H_1^\perp(z, z^2 \boldsymbol{k}_T^2)\, \frac{\not{k}_T}{M_h}\right) \mathcal{P}_-\,, \tag{2.52}$$

[3]Note that on the way to the final state hadrons, the color carried by the initial quark can be neutralized without breaking factorization, for instance via soft gluon contributions.

where we introduced the *parton fragmentation functions*

$$D_1(z, z^2\boldsymbol{k}_T^2) = \frac{1}{2z}\int \mathrm{d}k^2\, \mathrm{d}(2k\cdot P_h)\, \delta\left(\boldsymbol{k}_T^2 + \frac{M_h^2}{z^2} + k^2 - \frac{2k\cdot P_h}{z}\right)\left[B_2 + \frac{1}{z}B_3\right], \tag{2.53a}$$

$$H_1^\perp(z, z^2\boldsymbol{k}_T^2) = \frac{1}{2z}\int \mathrm{d}k^2\, \mathrm{d}(2k\cdot P_h)\, \delta\left(\boldsymbol{k}_T^2 + \frac{M_h^2}{z^2} + k^2 - \frac{2k\cdot P_h}{z}\right)[-B_4]\,. \tag{2.53b}$$

The fragmentation function $H_1^\perp$ is known with the name of *Collins function* [46].

The individual fragmentation functions can be isolated by means of the projection[4]

$$\Delta^{[\Gamma]} \equiv \mathrm{Tr}\,(\Delta\Gamma)\,, \tag{2.54}$$

where Γ stands for a specific Dirac structure. In particular, we see that

$$D_1(z, z^2\boldsymbol{k}_T^2) = \frac{1}{2}\,\Delta^{[\gamma^-]}(z, \boldsymbol{k}_T), \tag{2.55a}$$

$$\frac{\epsilon_T^{ij} k_{T\,j}}{M_h} H_1^\perp(z, z^2\boldsymbol{k}_T^2) = \Delta^{[\mathrm{i}\sigma^{i-}\gamma_5]}(z, \boldsymbol{k}_T). \tag{2.55b}$$

As we have done with the distribution functions, it will be helpful to introduce the notation

$$D^{(1/2)}(z, z^2\boldsymbol{k}_T^2) \equiv \frac{|\boldsymbol{k}_T|}{2M_h}\, D(z, z^2\boldsymbol{k}_T^2), \tag{2.56a}$$

$$D^{(n)}(z, z^2\boldsymbol{k}_T^2) \equiv \left(\frac{\boldsymbol{k}_T^2}{2M_h^2}\right)^n D(z, z^2\boldsymbol{k}_T^2), \tag{2.56b}$$

for n integer.

2.4.1 Correlation function in helicity formalism

Expressing the Dirac matrices of Eq. (2.52) in the chiral or Weyl representation, as done in Sec. 2.1.1 on page 15, we get for the leading twist part of the correlation function the expression

$$\left(\mathcal{P}_-\,\Delta(z, \boldsymbol{k}_T)\,\gamma^-\right)_{kl} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & D_1(z, z^2\boldsymbol{k}_T^2) & \mathrm{i}e^{\mathrm{i}\phi_k}\,\frac{|\boldsymbol{k}_T|}{M_h}\,H_1^\perp(z, z^2\boldsymbol{k}_T^2) & 0 \\ 0 & -\mathrm{i}e^{-\mathrm{i}\phi_k}\,\frac{|\boldsymbol{k}_T|}{M_h}\,H_1^\perp(z, z^2\boldsymbol{k}_T^2) & D_1(z, z^2\boldsymbol{k}_T^2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.57}$$

Restricting ourselves to the subspace of good quark fields and using the chirality basis, we can rewrite the correlation function as

$$D(z, \boldsymbol{k}_T)_{\chi_2'\chi_2} = \begin{pmatrix} D_1(z, z^2\boldsymbol{k}_T^2) & \mathrm{i}e^{\mathrm{i}\phi_k}\,\frac{|\boldsymbol{k}_T|}{M_h}\,H_1^\perp(z, z^2\boldsymbol{k}_T^2) \\ -\mathrm{i}e^{-\mathrm{i}\phi_k}\,\frac{|\boldsymbol{k}_T|}{M_h}\,H_1^\perp(z, z^2\boldsymbol{k}_T^2) & D_1(z, z^2\boldsymbol{k}_T^2) \end{pmatrix}. \tag{2.58}$$

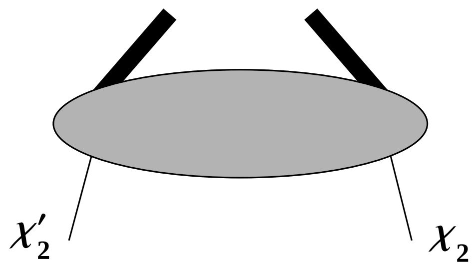


Figure 2.2. Illustration of the position of the indices of the correlation matrix.

Fig. 2.2 shows diagrammatically the position of the indices of the correlation function. Besides being Hermitean, the matrix fulfills the properties of angular momentum conservation and parity invariance. Because of the presence of factors $\mathrm{e}^{\mathrm{i}l\phi_k}$, we have to take into account l units of angular momentum in the initial state, therefore the condition of angular momentum conservation is $\chi_2' = \chi_2 + l$ and the condition of parity invariance is

$$D(z, \boldsymbol{k}_T)_{\chi_2'\chi_2} = (-1)^l \, D(z, \boldsymbol{k}_T)_{-\chi_2' \, -\chi_2}\Big|_{l\to -l}. \tag{2.59}$$

Positivity of the correlation matrix implies the bounds

$$D_1(z, z^2\boldsymbol{k}_T^2) \geq 0, \tag{2.60a}$$

$$\left|H_1^{\perp(1)}(z, z^2\boldsymbol{k}_T^2)\right| \leq D_1^{(1/2)}(z, z^2\boldsymbol{k}_T^2). \tag{2.60b}$$

2.5 Structure functions

Inserting the parameterizations of the different correlators in the expression (1.58) of the hadronic tensor and contracting it with the leptonic tensor, one can calculate the leptoproduction cross section for semi-inclusive DIS and project out the different structure functions appearing in Eq. 1.32. To have a compact notation for the results, we introduce the notation

$$\mathcal{C}[w f D] = x_B \sum_q e_q^2 \int d^2\boldsymbol{p}_T \, d^2\boldsymbol{k}_T \, \delta^{(2)}(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z) \, w(\boldsymbol{p}_T, \boldsymbol{k}_T) \, f^q(x_B, p_T^2) \, D^q(z, k_T^2), \tag{2.61}$$

with the unit vector $\hat{\boldsymbol{h}} = \boldsymbol{P}_{h\perp}/|\boldsymbol{P}_{h\perp}|$, where $w(\boldsymbol{p}_T, \boldsymbol{k}_T)$ is an arbitrary function.

[4]The absence of the factor 1/2 in Eq. (2.54) as compared to Eq. (2.24) is due to the absence of an averaging over initial states.

These are the expressions for the structure functions appearing in Eq. (1.32)

$$F_{UU,T} = C[f_1 D_1], \qquad F_{UU,L} = 0, \qquad F_{UU}^{\cos\phi_h} = 0 \tag{2.62}$$

$$F_{UU}^{\cos 2\phi_h} = C\Big[-\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T) - \boldsymbol{k}_T\cdot\boldsymbol{p}_T}{M M_h} h_1^{\perp} H_1^{\perp}\Big], \tag{2.63}$$

$$F_{LU}^{\sin\phi_h} = 0, \qquad F_{UL}^{\sin\phi_h} = 0, \tag{2.64}$$

$$F_{UL}^{\sin 2\phi_h} = C\Big[-\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T) - \boldsymbol{k}_T\cdot\boldsymbol{p}_T}{M M_h} h_{1L}^{\perp} H_1^{\perp}\Big], \tag{2.65}$$

$$F_{LL} = C[g_{1L} D_1], \qquad F_{LL}^{\cos\phi_h} = 0, \tag{2.66}$$

$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = C\Big[-\frac{\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\Big], \qquad F_{UT,L}^{\sin(\phi_h-\phi_S)} = 0, \tag{2.67}$$

$$F_{UT}^{\sin(\phi_h+\phi_S)} = C\Big[-\frac{\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{k}_T}{M_h} h_1 H_1^{\perp}\Big], \tag{2.68}$$

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = C\Big[\frac{2\,(\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{p}_T)\,(\boldsymbol{p}_T\cdot\boldsymbol{k}_T) + \boldsymbol{p}_T^2\,(\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{k}_T) - 4\,(\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{p}_T)^2\,(\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{k}_T)}{2M^2 M_h} h_{1T}^{\perp} H_1^{\perp}\Big], \tag{2.69}$$

$$F_{UT}^{\sin\phi_S} = 0, \qquad F_{UT}^{\sin(2\phi_h-\phi_S)} = 0, \tag{2.70}$$

$$F_{LT}^{\cos(\phi_h-\phi_S)} = C\Big[\frac{\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{p}_T}{M} g_{1T} D_1\Big], \qquad F_{LT}^{\cos\phi_S} = 0, \qquad F_{LT}^{\cos(2\phi_h-\phi_S)} = 0, \tag{2.71}$$

It has to be stressed that in much of the past literature a different definition of the azimuthal angles has been used, whereas in the present work we adhere to the Trento conventions [22]. To compare with old papers, the signs of ϕ_h and of ϕ_S have to be reversed. All the nonzero structure functions here are consistent with those given in Eqs. (36) and (37) of Ref. [32] when only photon exchange is taken into consideration.

2.6 Weighted asymmetries

In the structure functions we have expressions such as

$$C\Big[-\frac{\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\Big] \tag{2.72}$$

where the two functions appear in a convolution. The general way to split the convolution is to use transverse-momentum-weighted asymmetries. For instance

$$W = \int d^2\boldsymbol{P}_{h\perp}\,\frac{|\boldsymbol{P}_{h\perp}|}{M}\,C\Big[-\frac{\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\Big] \tag{2.73}$$

in fact

$$\begin{aligned} W &= -x_B \sum_q e_q^2 \int d^2\boldsymbol{P}_{h\perp}\, d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^{(2)}(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z_h)\, \frac{\boldsymbol{P}_{h\perp}\cdot \boldsymbol{p}_T}{M}\, f_{1T}^{\perp q} D_1^q, \\ &= -x_B\, z_h^2 \sum_q e_q^2 \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \frac{(\boldsymbol{p}_T - \boldsymbol{k}_T)\cdot \boldsymbol{p}_T}{M^2}\, f_{1T}^{\perp q} D_1^q, \\ &= -x_B \sum_q e_q^2 \int d^2\boldsymbol{p}_T\, \frac{|\boldsymbol{p}_T|^2}{M^2}\, f_{1T}^{\perp q}\, z_h^2 \int d^2\boldsymbol{p}_T\, D_1^q \\ &= -x_B \sum_q e_q^2\, 2\, f_{1T}^{\perp(1)q}(x_B)\, D_1(z_h) \end{aligned} \tag{2.74}$$

2.7 Leading twist part and connection with helicity formalism

To identify the leading twist contributions to the cross section, it is convenient to define the projectors

$$\mathcal{P}_+ = \frac{1}{2}\,\gamma^-\gamma^+, \qquad\qquad \mathcal{P}_- = \frac{1}{2}\,\gamma^+\gamma^-. \tag{2.75}$$

Before the interaction with the virtual photon, the relevant components of the quark fields are the plus components, $\psi_+ = \mathcal{P}_+\,\psi$. They are usually referred to as the *good components*. Vice versa, after the interaction with the virtual photon, the relevant components of the outgoing quark fields are the minus components, $\psi_- = \mathcal{P}_-\,\psi$. Therefore, the leading twist part of the hadronic tensor in Eq. (1.25) can be projected out in the following way [5]

$$\begin{aligned} 2MW^{\mu\nu}(q,P,S,P_h) &\approx 4z_h\, \mathcal{I}\Big[\mathrm{Tr}(\mathcal{P}_+\,\Phi(x_B,\boldsymbol{p}_T,S)\,\mathcal{P}_-\,\gamma^\mu\,\mathcal{P}_-\,\Delta(z_h,\boldsymbol{k}_T)\,\mathcal{P}_+\,\gamma^\nu)\Big] \\ &= 4z_h\,\mathcal{I}\left[\mathrm{Tr}\left(\mathcal{P}_+\,\Phi(x_B,\boldsymbol{p}_T,S)\,\gamma^+\,\frac{\gamma^-\gamma^\mu}{2}\,\mathcal{P}_-\,\mathcal{P}_-\,\Delta(z_h,\boldsymbol{k}_T)\,\gamma^-\,\frac{\gamma^+\gamma^\nu}{2}\,\mathcal{P}_+\right)\right]. \end{aligned} \tag{2.76}$$

The differential cross section becomes

$$\begin{aligned} &\frac{\mathrm{d}^6\sigma}{\mathrm{d}x_B\,\mathrm{d}y\,\mathrm{d}z_h\,\mathrm{d}\phi_S\,\mathrm{d}^2\boldsymbol{P}_{h\perp}} \\ &\approx \sum_q \frac{\alpha^2 e_q^2}{s x_B Q^2}\, L_{\mu\nu}(l,l',\lambda_e)\, \mathcal{I}\left[\mathrm{Tr}\left(\mathcal{P}_+\,\Phi(x_B,\boldsymbol{p}_T,S)\,\gamma^+\,\frac{\gamma^-\gamma^\mu}{2}\,\mathcal{P}_-\,\mathcal{P}_-\,\Delta(z_h,\boldsymbol{k}_T)\,\gamma^-\,\frac{\gamma^+\gamma^\nu}{2}\,\mathcal{P}_+\right)\right] \\ &\approx \sum_q \mathcal{I}\left[\left(\mathcal{P}_+\,\Phi(x_B,\boldsymbol{p}_T,S)\gamma^+\right)_{ij} \frac{\alpha^2 e_q^2}{s x_B Q^2}\, L_{\mu\nu}(l,l',\lambda_e) \left(\frac{\gamma^-\gamma^\mu}{2}\,\mathcal{P}_-\right)_{jl} \left(\frac{\gamma^+\gamma^\nu}{2}\,\mathcal{P}_+\right)_{mi} \left(\mathcal{P}_-\,\Delta(z_h,\boldsymbol{k}_T)\,\gamma^-\right)_{lm}\right]. \end{aligned} \tag{2.77}$$

[5] Note that $(\bar{\psi}_+) = \bar{\psi}\,\mathcal{P}_-$ and $(\bar{\psi}_-) = \bar{\psi}\,\mathcal{P}_+$.

In Sec. 2.1.1 on page 15, we have seen in detail how the insertion of the projectors effectively reduces the four-dimensional Dirac space into a two-dimensional subspace. Chiral-right and chiral-left good quark spinors can be used as a basis in this space. Therefore, it is possible to replace the Dirac indices with chirality indices (of good fields). By doing this, we put particular evidence on the connection with the helicity/chirality formalism (see e.g. Refs. 61 and 9). Writing all the components of the cross section in the chirality space of the good fields, we obtain

$$\begin{aligned}\frac{d^6\sigma}{dx_B\,dy\,dz_h\,d\phi_S\,d^2\boldsymbol{P}_{h\perp}} &= \rho(S)_{\Lambda_1\Lambda_1'}\ \mathcal{I}\left[\left(\mathcal{P}_+\,\Phi(x_B,\boldsymbol{p}_T)\gamma^+\right)^{\Lambda_1'\Lambda_1}_{\chi_1'\chi_1}\Big(\frac{d\sigma^{eq}}{dy}\Big)^{\chi_1\chi_1';\chi_2\chi_2'}\left(\mathcal{P}_-\,\Delta(z_h)\,\gamma^-\right)_{\chi_2'\chi_2}\right]\\ &\equiv \rho(S)_{\Lambda_1\Lambda_1'}\ \mathcal{I}\left[F(x_B,\boldsymbol{p}_T)^{\Lambda_1'\Lambda_1}_{\chi_1\chi_1'}\Big(\frac{d\sigma^{eq}}{dy}\Big)^{\chi_1\chi_1';\chi_2\chi_2'}D(z_h,\boldsymbol{k}_T)_{\chi_2'\chi_2}\right].\end{aligned} \tag{2.78}$$

The elementary electron-quark scattering matrix is

$$\begin{aligned}\Big(\frac{d\sigma^{eq}}{dy}\Big)^{\chi_1\chi_1';\chi_2\chi_2'} &= \frac{\alpha^2 e_q^2}{s x_B Q^2}\,L_{\mu\nu}(l,l',\lambda_e)\left(\frac{\gamma^-\gamma^\mu}{2}\,\mathcal{P}_-\right)^{\chi_1\chi_2}\left(\frac{\gamma^+\gamma^\nu}{2}\,\mathcal{P}_+\right)^{\chi_2'\chi_1'}\\ &= \frac{2\alpha^2 e_q^2}{s x_B y^2}\left(\begin{array}{cc|cc} A(y)+\lambda_e C(y) & 0 & 0 & -B(y)\\ 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ -B(y) & 0 & 0 & A(y)-\lambda_e C(y)\end{array}\right),\end{aligned} \tag{2.79}$$

where

$$A(y) = 1-y+\frac{y^2}{2}, \qquad B(y) = (1-y), \qquad C(y) = y\left(1-\frac{y}{2}\right). \tag{2.80}$$

The internal blocks have indices $\chi_1'\,\chi_1$ and the outer matrix has indices $\chi_2'\,\chi_2$.

3

Spectator model: a workbench

There are many models used for the calculation of parton distribution functions (and also fragmentation functions). We are here going to take into consideration one, the so-called *spectator model* because it allows us to compute in a relatively easy way transverse momentum dependent distribution and fragmentation functions and discuss some interesting features.

3.1 Basics

In the spectator model, the proton is supposed to be coupled to a quark and a spectator through some sort of effective vertes. Therefore, the sum of all possible intermediate states in the definition of the correlator is effectively replaced by a single, on-shell state: the spectator. For a proton target, the spectator has the quantum numbers of a diquark. Conservation of angular momentum requires the diquark to have spin 0 (scalar diquark) or 1 (vector diquark). From a flavor point of view, it could be a isoscalar (*ud*) or a isovector (*uu*).

The correlation function (1.21) becomes

$$
\begin{aligned}
\Phi(k,P,S) &= \sum_X \int \frac{d^3\boldsymbol{P}_X}{(2\pi)^3\, 2P_X^0}\, \langle P,S\,|\,\bar{\psi}(0)\,|X\rangle\langle X|\,\psi(0)\,|P,S\,\rangle\, \delta^{(4)}\Big(P-k-P_X\Big)\\
&\to \int \frac{d^4P_X}{(2\pi)^4}\,(2\pi)\,\delta(P_X^2-M_X^2)\,\theta(P_X^+)\,\delta^{(4)}(P_X-(P-k))\,\langle P,S|\bar{\psi}(0)|P_X\rangle\,\langle P_X|\psi(0)|P,S\,\rangle\\
&= \frac{1}{(2\pi)^3}\,\delta((P-k)^2-M_s^2)\,\theta((P-k)^+)\,\langle P,S|\bar{\psi}(0)|P-k\,\rangle\,\langle P-k\,|\psi(0)|P,S\,\rangle\ .
\end{aligned}
$$

We want to compute $\Phi(x,\boldsymbol{k}_T,S) = \int dk^-\ \Phi(k,P,S)$. We use the δ function to perform the k^- integration

$$
\begin{aligned}
\delta\Big((P-k)^2-M_s^2\Big) &= \delta\Big(2(P-k)^+(P-k)^- - \boldsymbol{k}_T^2 - M_s^2\Big)\\
&= \delta\Big(-(1-x)(2P^+k^- + M^2) - \boldsymbol{k}_T^2 - M_s^2\Big)\\
&= \frac{1}{2(1-x)P^+}\,\delta\Big(k^- + \frac{\boldsymbol{k}_T^2 + M_s^2 + (1-x)M^2}{2(1-x)P^+}\Big)
\end{aligned}
\tag{3.1}
$$

so that

$$
\begin{aligned}
\Phi(x,\boldsymbol{k}_T,S) &= \int dk^-\ \Phi(k,P,S)\Big|_{k^+=xP^+}\\
&= \frac{1}{(2\pi)^3}\,\frac{1}{2(1-x)P^+}\,\langle P,S|\bar{\psi}(0)|P-k\,\rangle\,\langle P-k\,|\psi(0)|P,S\,\rangle\Big|_{\substack{k^+=x\,P^+,\\ k^2=-(\boldsymbol{k}_T^2+L^2)/(1-x)-m^2}}\ ,
\end{aligned}
\tag{3.2}
$$

where we introduce for convenience the function

$$
L^2 = x(1-x)\Big(-M^2 + \frac{m^2}{x} + \frac{M_s^2}{(1-x)}\Big). \tag{3.3}
$$

For simplicity, we shall consider only the scalar diquark. In this case, the $P \to q(2q)$ "scattering amplitude" at tree level can be written

$$
\langle P-k|\psi(0)|P,S\,\rangle = \frac{i}{\not{k}-m}\,\Upsilon_s\,U(P,S) = \mathrm{i}\,\frac{\not{k}+m}{k^2-m^2}\ \mathrm{i}g_s(k^2)\,\frac{1+\gamma_5\not{S}}{2}\,U(P,S)\ . \tag{3.4}
$$

Note that this is not really a simple scattering amplitude, since the quark propagator is included. It follows that

$$
\langle P,S\,|\,\bar{\psi}(0)\,|P-k\rangle = \langle P-k|\,\psi(0)\,|P,S\,\rangle^\dagger\gamma^0 = \bar{U}(P,S)\frac{1+\gamma_5\not{S}}{2}\,(-\mathrm{i})\,g_s\,(-\mathrm{i})\,\frac{\not{k}+m}{k^2-m^2}. \tag{3.5}
$$

The correlation function becomes

$$
\Phi(x,\boldsymbol{k}_T,S) = \frac{1}{(2\pi)^3}\,\frac{1}{2(1-x)P^+}\,\frac{g_s^2(k^2)}{(k^2-m^2)^2}\Big[(\not{k}+m)\,\frac{1+\gamma_5\not{S}}{2}\,(\not{P}+M)\,(\not{k}+m)\Big]\Big|_{\ldots} \tag{3.6}
$$

3.2 Calculation of f_1

Now we can compute all kinds of distribution functions we want in a straightforward way. Let's start with f_1, then we simply calculate (see, e.g., Ref. [31])

$$\begin{aligned}\Phi^{[\gamma^+]} &= \frac{1}{2}\frac{1}{(2\pi)^3}\frac{1}{2(1-x)P^+}\frac{g_s^2}{(k^2-m^2)^2}\mathrm{Tr}\Big[(\not{k}+m)\,\frac{1+\gamma_5\not{S}}{2}\,(\not{P}+M)\,(\not{k}+m)\gamma^+\Big]\\ &= \frac{g_s^2}{16\pi^3}\frac{m(m+2Mx)-k^2+2x\,k\cdot P}{(1-x)\,(k^2-m^2)^2}\\ &= \frac{g_s^2}{16\pi^3}\frac{1-x}{(k_T^2+L^2)^2}\Big(k_T^2+(m+Mx)^2\Big).\end{aligned} \tag{3.7}$$

There are now a couple of things to observe: first, the result does not depend on the spin, meaning that the Sivers function is zero in this model. Second, as it is the function cannot be integrated over k_T: it would give rise to a logarithmic divergence. We can overcome this problem in two ways: instead of a coupling constant, we can use a form factor $g_s^2(k^2)$ that suppresses the hight-k_T tails; alternatively, we can choose a cutoff Λ and stop the integration there. Let's follow the second option, since it is simpler. The result of the integration is

$$f_1(x) = \frac{g_s^2\,(1-x)}{8\pi^2}\bigg(x\,\frac{\Lambda^2}{L^2}\,\frac{(m+M)^2-M_s^2}{L^2+\Lambda^2}+\ln\frac{L^2+\Lambda^2}{\Lambda^2}\bigg). \tag{3.8}$$

Ex. 3.1

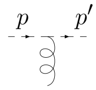
Calculate the transversity function using the same procedure as for f_1.

3.3 Calculation of the Sivers function

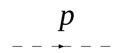
It is not surprising that the previous calculation led to a vanishing Sivers function. In order to have a nonzero T-odd function, we need the interference between two scattering amplitudes with different imaginary parts. At tree level, we have just one real scattering amplitude.

A good way to introduce imaginary parts is to insert some loops. The optical theorem tells you that if a scattering amplitude can be seen as the cut diagram of some other process, then it means that it has an imaginary part. In other words, you have an imaginary part whenever you can cut the amplitude by putting some internal particles on shell. It's impossible to do this if you have a tree-level diagram, but it might be possible for amplitudes with loops. For instance, the self-energy diagram...

Let'suppose that quark and diquark in our model can interact through the exchange of some vector boson. We know that they do interact via the exchange of gluons. For the moment being, let's suppose they interact via an Abelian gluon (i.e., with no color structure). The gluon-quark coupling is the standard one (without the color matrix). For the gluon-diquark coupling and for the scalar propagators, we take the typical coupling between a photon and a scalar particle and a typical scalar propagator, i.e.



$$= -\mathrm{i}e_c\,(p+p')^\mu$$

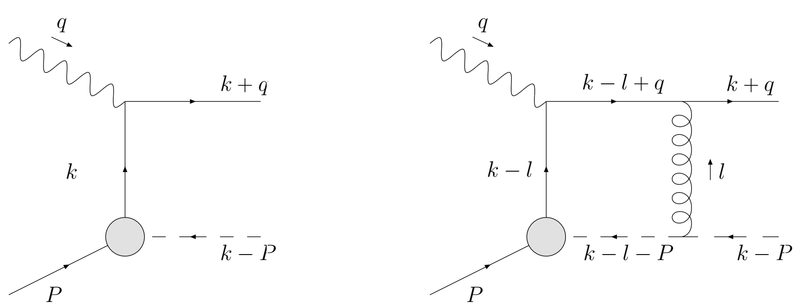


$$= \frac{\mathrm{i}}{p^2 - M_S^2 + \mathrm{i}\epsilon} \tag{3.9}$$

The charges of the two objects have to be equal (and opposite), so that the proton can be color neutral.

There are three possible places to draw gluon loops, but none of them can be cut, meaning that none of them can give rise to the imaginary part we need. The situation before 2002 was that T-odd functions could not be produced with this model. However, it turned out that we had forgotten an important piece in the calculation of the distribution functions: the gauge link or Wilson line. We'll come back more formally to this issue, but now let's take a more pragmatical point of view and let's take into consideration the diagram shown in Fig. 3.1 b.

The gluon in this case couples to the quark after the scattering with the photon. This is a potentially dangerous thing to do, because in the formalism so far we carefully avoided taking gluons into account and pretended that only the quark-photon interaction was relevant at leading twist in the low-transverse-momentum regime. It turns out nevertheless that at leading twist, this diagram can be completely reabsorbed into the quark-quark correlation function. Let's see how it works.



(a) (b)

Figure 3.1.

The amplitude at tree level, including now also the photon interaction, corresponds to (Fig. 3.1 a)

$$\mathcal{M}^{(0)} = \bar{u}(k+q)\,(-\mathrm{i}e_q)\,\not\epsilon(q)\,\frac{\mathrm{i}(\not k + m)}{k^2 - m^2}\;\mathrm{i}g_s(k^2)\,\frac{1+\gamma_5\not S}{2}\,U(P,S) \tag{3.10}$$

We write down the amplitude with the loop, Fig. 3.1 b. Two important things: we use Feynman gauge for the gluon propagator and we also keep track of the iϵ terms in the propagators.

$$
\begin{aligned}
\mathcal{M}^{(1)} = \bar{u}(k+q) \int \frac{d^4 l}{(2\pi)^4} (-\mathrm{i}e_c\gamma_\mu) \frac{-\mathrm{i}g^{\mu\nu}}{[l^2 - m_g^2 + \mathrm{i}\epsilon]} (-\mathrm{i}e_c) \frac{(2k-l-2P)_\nu\, \mathrm{i}}{[(k-l-P)^2 - M_s^2 + \mathrm{i}\epsilon]} \\
\times \frac{\mathrm{i}(\not{k} + \not{q} - \not{l} + m)}{[(k+q-l)^2 - m^2 + \mathrm{i}\epsilon]} (-\mathrm{i}e_q)\not{\epsilon}(q) \frac{\mathrm{i}(\not{k} - \not{l} + m)}{[(k-l)^2 - m^2 + \mathrm{i}\epsilon]}\, \mathrm{i}g_s((k-l)^2)\, \frac{1+\gamma_5 \not{S}}{2}\, U(P,S)
\end{aligned}
\tag{3.11}
$$

Now, we perform a simplification that goes under the name of "eikonal approximation" and consists simply in taking into account only the leading parts of the momenta of the quark after the photon scattering. As we know, the $-$ components are the leading ones. Therefore, the quark propagator in the upper part of the diagram becomes

$$
\frac{\mathrm{i}(\not{k} + \not{q} - \not{l} + m)}{(k+q-l)^2 - m^2 + \mathrm{i}\epsilon} \approx \mathrm{i}\, \frac{(k+q)^-\gamma^+}{-2l^+(k+q)^- + \mathrm{i}\epsilon} = \frac{\mathrm{i}}{2}\, \frac{\gamma^+}{-l^+ + \mathrm{i}\epsilon} \tag{3.12}
$$

In the last step it is essential that to have $(k+q)^- \geq 0$. This condition is guaranteed by the fact that we want to have an outgoing quark with momentum $k+q$ in the final state. The above expression is often referred to as an *eikonal propagator*.

If we insert the simplified propagator in the scattering amplitude and multiply it by the (conjugate) of the tree-level amplitude we obtain

$$
\begin{aligned}
\bar{\mathcal{M}}^{(0)}\mathcal{M}^{(1)} = {} & (-\mathrm{i})g_s^*(k^2)\, \frac{-\mathrm{i}(\not{k}+m)}{k^2 - m^2}\, \mathrm{i}e_q\not{\epsilon}^*\, u(k+q)\bar{u}(k+q) \\
& \int \frac{d^4 l}{(2\pi)^4} (-\mathrm{i}e_c\gamma^\nu) \frac{-\mathrm{i}}{[l^2 - m_g^2 + \mathrm{i}\epsilon]} (-\mathrm{i}e_c) \frac{(2k-l-2P)_\nu\, \mathrm{i}}{[(k-l-P)^2 - M_s^2 + \mathrm{i}\epsilon]}\, \frac{\mathrm{i}}{2}\, \frac{\gamma^+}{-l^+ + \mathrm{i}\epsilon} \\
& \times (-\mathrm{i}e_q)\not{\epsilon}(q)\, \frac{\mathrm{i}(\not{k} - \not{l} + m)}{[(k-l)^2 - m^2 + \mathrm{i}\epsilon]}\, \mathrm{i}g_s((k-l)^2)\, \frac{1+\gamma_5\not{S}}{2}\, U(P,S)\bar{U}(P,S)
\end{aligned}
\tag{3.13}
$$

The crucial observation at this point is that in the spirit of the eikonal approximation, the only possibility to have a nonzero result is if $\gamma^\nu = \gamma^-$

$$
\ldots(\not{k}+\not{q}+m)\gamma^\nu\gamma^+\ldots \approx \ldots(k+q)^-\gamma^+\gamma^\nu\gamma^+\ldots = \ldots(k+q)^-\gamma^+\gamma^-\gamma^+\ldots = \ldots 2(k+q)^-\gamma^+\ldots \tag{3.14}
$$

We can then rewrite

$$
\begin{aligned}
\bar{\mathcal{M}}^{(0)}\mathcal{M}^{(1)} = {} & \frac{e_q^2\, e_c^2\, g_s^*(k^2)}{k^2 - m^2}\, \not{\epsilon}^*(q)\, (\not{k} + \not{q} + m)\, \not{\epsilon}(q) \\
& \int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{i}\, g_s((k-l)^2)\, (2k-l-2P)^+\, (\not{k} - \not{l} + m)\, (1+\gamma_5\not{S})\, (\not{P} + m)\, (\not{k} + m)}{2\, [l^2 - m_g^2 + \mathrm{i}\epsilon]\, [-l^+ + \mathrm{i}\epsilon]\, [(k-l)^2 - m^2 + \mathrm{i}\epsilon][(k-l-P)^2 - M_s^2 + \mathrm{i}\epsilon]}
\end{aligned}
\tag{3.15}
$$

We can associate a modified Feynman diagram to the amplitude $\mathcal{M}^{(1)}$ after the eikonal approximation. It is drawn in Fig. 3.2. In other words, we managed to "factorize" the photon-quark interaction from the rest. In fact, the previous equation can be written as

$$
\bar{\mathcal{M}}^{(0)}\mathcal{M}^{(1)} = e_q^2\, \mathrm{Tr}\Big[\not{\epsilon}^*(q)\, (\not{k} + \not{q} + m)\not{\epsilon}(q)\, \langle P,S|\bar{\psi}(0)|P-k\rangle^{(1)}\, \langle P-k|\psi(0)|P,S\rangle^{(0)}\Big] \tag{3.16}
$$

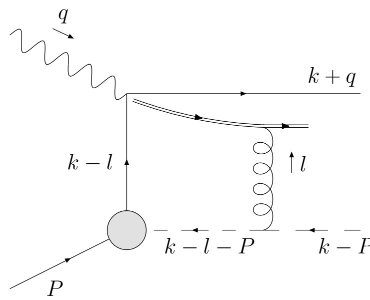


Figure 3.2.

We could now forget about the photon interaction and focus only on the lower part of the diagram, which now describes our $P \to q(2q)$ amplitude at the one-loop level, i.e., including the modification due to (the leading-twist part of) the gluon-quark final state interaction. The Feynman diagram associated with this amplitude is depicted in Fig. 3.3. Whenever we meet the eikonal line in such

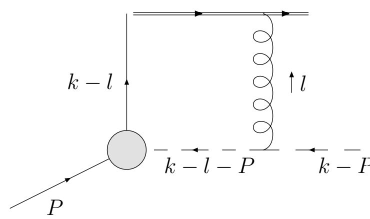
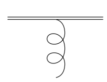
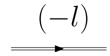
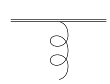


Figure 3.3.

a Feynman diagram, we have to use the following Feynman rules

 $= -\mathrm{i} e_c \, n_-^\mu$ $(-l)$  $= \dfrac{\mathrm{i}}{-l \cdot n_- + \mathrm{i}\epsilon}$ (3.17)

If we properly take into account the color structure of the gluon-eikonal vertex, it becomes

 $= \mathrm{i} g \, t^a \, n_-^\mu$ (3.18)

Note that the modifications we discussed in the context of the eikonal approximation are totally independent of the specific form of the diquark model. However, they do depend crucially on two things: the gauge choice and whether the eikonal propagator represents an outgoing or incoming particle. We'll come back to this later.

Now, let's proceed with our model calculation. The quark-quark correlator at one loop reads

$$\Phi(x, \boldsymbol{k}_T, S)^{(1)} = \frac{1}{(2\pi)^3} \frac{1}{2(1-x)P^+} \frac{e_c^2 \, g_s^*(k^2)}{k^2 - m^2} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\mathrm{i}\, g_s((k-l)^2)\,(2k - l - 2P)^+ \,(\not{k} - \not{l} + m)\,(1 + \gamma_5 \not{S})\,(\not{P} + m)\,(\not{k} + m)}{2\,[l^2 - m_g^2 + \mathrm{i}\epsilon]\,[-l^+ + \mathrm{i}\epsilon]\,[(k-l)^2 - m^2 + \mathrm{i}\epsilon][(k - l - P)^2 - M_s^2 + \mathrm{i}\epsilon]} + \mathrm{H.c.} \tag{3.19}$$

Since we want to compute the Sivers function, we consider only the part depending on the spin of the target and we perform the required trace (I use here the shorthand notation $\epsilon_T^{kS} = \epsilon_T^{\rho\sigma} k_{T\rho} S_{T\sigma}$)

$$-\frac{\epsilon_T^{kS}}{M} f_{1T}^{\perp}(x, k_T^2) = \frac{1}{2}\Big[\Phi^{[\gamma^+]}(x, k_T^2, S) - \Phi^{[\gamma^+]}(x, k_T^2, -S)\Big] \tag{3.20}$$

Let's first look at the result of the Dirac trace

$$\begin{aligned}\frac{1}{2}\,\mathrm{Tr}\Big[(\not k - \not l + m)\,\gamma_5 \not S\,(\not P + m)\,(\not k + m)\,\not n_-\Big] &= 2\mathrm{i}\left(M\epsilon^{k l n_- S} + m\epsilon^{l n_- P S}\right)\\ &= -2\mathrm{i}\left[(xM + m)P^+\epsilon_T^{lS} - M\,l^+\epsilon_T^{kS} + MS^+\epsilon^{kl}\right]\end{aligned} \tag{3.21}$$

The last term cannot give a contribution because it will never match with the prefactor of the Sivers function. Mathematically, this is due to the fact that once the integration over l is carried out, the only transverse vector we have at our disposal is k_T. Therefore, ϵ^{kl} after the integration can only be proportional to ϵ^{kk}, which is zero.

Using this result we write

$$\begin{aligned}-\frac{\epsilon_T^{kS}}{M} f_{1T}^{\perp}(x, k_T^2) &= \frac{1}{(2\pi)^3}\,\frac{1}{2(1-x)P^+}\,\frac{e_c^2\, g_s^*(k^2)}{k^2 - m^2}\\ &\quad \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,\frac{g_s((k-l)^2)\,(2k - l - 2P)^+\,\big[(xM + m)P^+\epsilon_T^{lS} - M\,l^+\epsilon_T^{kS}\big]}{[l^2 - m_g^2 + \mathrm{i}\epsilon]\,[-l^+ + \mathrm{i}\epsilon]\,[(k-l)^2 - m^2 + \mathrm{i}\epsilon][(k - l - P)^2 - M_s^2 + \mathrm{i}\epsilon]} + \text{c.c.}\\ &= \frac{1}{(2\pi)^3}\,\frac{1}{2(1-x)P^+}\,\frac{e_c^2\, g_s^*(k^2)}{k^2 - m^2}\\ &\quad 2\,\mathrm{Re}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,\frac{g_s((k-l)^2)\,(2k - l - 2P)^+\,\big[(xM + m)P^+\epsilon_T^{lS} - M\,l^+\epsilon_T^{kS}\big]}{[l^2 - m_g^2 + \mathrm{i}\epsilon]\,[-l^+ + \mathrm{i}\epsilon]\,[(k-l)^2 - m^2 + \mathrm{i}\epsilon][(k - l - P)^2 - M_s^2 + \mathrm{i}\epsilon]}\end{aligned} \tag{3.22}$$

In the last step, we assumed that form factor is real. We came to a very important point, namely that the Sivers function is proportional to the real part of that long integral. It turns out that a nonzero real part is obtained only if we pick up the two poles from the $[-l^+ + \mathrm{i}\epsilon]$ and $[(k-l-P)^2 - M_s^2 + \mathrm{i}\epsilon]$ denominators. Picking up any other pole leads to a vanishing result. Note that we knew the Sivers function had to be proportional to the imaginary part of the one-loop scattering amplitude. The result we obtained is consistent with Cutkosky's rules, which state that the imaginary part of a Feynman diagram can be obtained by cutting the diagram in all possible ways such that the cut propagators can be put simultaneously on shell. The only possibility for the present diagram is in fact to cut the eikonal propagator and the spectator propagator.

Using the residue theorem and picking up the residue of a propagator pole is equivalent to replacing the propagator with a delta function, e.g.,

$$\frac{1}{[l^+ - \mathrm{i}\epsilon]} \to 2\pi\mathrm{i}\,\delta[l^+], \qquad \frac{1}{[(k - l - P)^2 - M_s^2 + \mathrm{i}\epsilon]} \to -2\pi\mathrm{i}\,\delta[(k - l - P)^2 - M_s^2], \tag{3.23}$$

$$\frac{1}{[l^2 - m_g^2 + \mathrm{i}\epsilon]} \to -2\pi\mathrm{i}\,\delta[l^2 - m_g^2], \qquad \frac{1}{[(k-l)^2 - m^2 + \mathrm{i}\epsilon]} \to -2\pi\mathrm{i}\,\delta[(k-l)^2 - m^2]. \tag{3.24}$$

Only the first two δ functions can go together. Let's take for instance the first and third one. If $l^+ = 0$, then $l^2 - m_g^2 = -\boldsymbol{l}_T^2 - m_g^2$ cannot be zero, unless we take the special case of zero gluon mass and zero transverse momentum, in which case the denominator of the integral would anyway vanish.

Ex. 3.2

Check that the other combinations of δ functions are kinematically forbidden.

To summarize, we have

$$\begin{aligned}-\frac{\epsilon_T^{kS}}{M} f_{1T}^{\perp}(x, k_T^2) = &\frac{1}{(2\pi)^3} \frac{1}{2(1-x)P^+} \frac{e_c^2\, g_s^*(k^2)}{k^2 - m^2} (2k - 2P)^+ (xM + m)\, P^+ \epsilon_T^{\rho\sigma} S_\sigma (2k - 2P)^+ \\ &2 \int \frac{d^4 l}{(2\pi)^2} \frac{g_s((k-l)^2)\, l_\rho}{[l^2 - m_g^2 + i\epsilon]\,[(k-l)^2 - m^2 + i\epsilon]}\, \delta[l^+]\, \delta[(k - l - P)^2 - M_s^2].\end{aligned} \tag{3.25}$$

The last integral must be proportional to $k_{T\rho}$, because it is the only external transverse momentum we have. Therefore, introducing the integral

$$D_1' = \int \frac{d^4 l}{(2\pi)^2} \frac{g_s((k-l)^2)\, (l_T \cdot k_T)/(k_T \cdot k_T)}{[l^2 - m_g^2]\,[(k-l)^2 - m^2]}\, \delta[l^+]\, \delta[(k - l - P)^2 - M_s^2], \tag{3.26}$$

we have

$$\int \frac{d^4 l}{(2\pi)^2} \frac{g_s((k-l)^2)\, l_{T\rho}}{[l^2 - m_g^2 + i\epsilon]\,[(k-l)^2 - m^2 + i\epsilon]}\, \delta[l^+]\, \delta[(k - l - P)^2 - M_s^2] = D_1'\, k_{T\rho}, \tag{3.27}$$

$$-\frac{\epsilon_T^{kS}}{M} f_{1T}^{\perp}(x, k_T^2) = -\frac{1}{(2\pi)^3} \frac{e_c^2\, g_s^*(k^2)}{k^2 - m^2} (xM + m)\, \epsilon_T^{kS}\, P^+\, D_1', \tag{3.28}$$

$$f_{1T}^{\perp}(x, k_T^2) = \frac{1}{(2\pi)^3} \frac{e_c^2\, g_s^*(k^2)}{k^2 - m^2} (xM + m)\, M\, P^+\, D_1'. \tag{3.29}$$

To solve the D_1' integral, we have now to specify our form factor. As before, we choose now a pointlike form factor and we pull it out of the integral

$$f_{1T}^{\perp}(x, k_T^2) = \frac{1}{(2\pi)^3} \frac{e_c^2\, g_s^2}{k^2 - m^2} (xM + m)\, M\, P^+\, D_1. \tag{3.30}$$

where

$$D_1 = \int \frac{d^4 l}{(2\pi)^2} \frac{(l_T \cdot k_T)/(k_T \cdot k_T)}{[l^2 - m_g^2]\,[(k-l)^2 - m^2]}\, \delta[l^+]\, \delta[(k - l - P)^2 - M_s^2]. \tag{3.31}$$

We use the two δ functions to perform the l^+ and l^- integrations first. Using $l^+ = 0$, the second δ function becomes

$$\begin{aligned}\delta[(k - l - P)^2 - M_s^2] &= \delta[(P-k)^2 - M_s^2 + l^2 + 2l \cdot (P - k)] = \delta[-\boldsymbol{l}_T^2 + 2l^-(1-x)P^+ + 2\boldsymbol{l}_T \cdot \boldsymbol{k}_T] \\ &= \frac{1}{2(1-x)P^+} \delta\Big[l^- - \frac{\boldsymbol{l}_T^2 - 2\boldsymbol{l}_T \cdot \boldsymbol{k}_T}{2(1-x)P^+}\Big]\end{aligned} \tag{3.32}$$

Using $l^+ = 0$ and the value for l^- fixed by the second δ function, the denominators become

$$[l^2 - m_g^2] = -\boldsymbol{l}_T^2 - m_g^2, \tag{3.33}$$

$$\begin{aligned}[(k-l)^2 - m^2] &= k^2 + l^2 - 2k\cdot l - m^2 = -\boldsymbol{l}_T^2 - 2xP^+\left(\frac{\boldsymbol{l}_T^2 - 2\boldsymbol{l}_T\cdot\boldsymbol{k}_T}{2(1-x)P^+}\right) + 2\boldsymbol{k}_T\cdot\boldsymbol{l}_T + k^2 - m^2 \\ &= \frac{1}{(1-x)}\Big[-(1-x)\boldsymbol{l}_T^2 - x\boldsymbol{l}_T^2 + 2x\boldsymbol{l}_T\cdot\boldsymbol{k}_T + 2(1-x)\boldsymbol{k}_T\cdot\boldsymbol{l}_T + (1-x)(k^2-m^2)\Big] \\ &= \frac{1}{(1-x)}\Big[-\boldsymbol{l}_T^2 + 2\,\boldsymbol{l}_T\cdot\boldsymbol{k}_T + (1-x)(k^2-m^2)\Big] = \frac{1}{(1-x)}\Big[-(\boldsymbol{l}_T - \boldsymbol{k}_T)^2 - L^2\Big].\end{aligned} \tag{3.34}$$

We eventually have

$$D_1 = \frac{1}{2(1-x)P^+}\int\frac{\mathrm{d}^2\boldsymbol{l}_T}{(2\pi)^2}\frac{\boldsymbol{l}_T\cdot\boldsymbol{k}_T}{\boldsymbol{k}_T^2}\frac{(1-x)}{[\boldsymbol{l}_T^2 + m_g^2][(\boldsymbol{l}_T - \boldsymbol{k}_T)^2 + L^2]}. \tag{3.35}$$

Changing variable $l_T \to l'_T + k_T$

$$D_1 = \frac{1}{2P^+}\int\frac{\mathrm{d}^2\boldsymbol{l}'_T}{(2\pi)^2}\frac{(\boldsymbol{l}'_T + \boldsymbol{k}_T)\cdot\boldsymbol{k}_T}{\boldsymbol{k}_T^2}\frac{1}{[(\boldsymbol{l}'_T + \boldsymbol{k}_T)^2 + m_g^2][\boldsymbol{l}'^2_T + L^2]} \tag{3.36}$$

At this point, we set the gluon mass to zero and we introduce a Feynman parameter to solve the integral

$$\begin{aligned} D_1 &= \frac{1}{2P^+}\int\frac{\mathrm{d}^2\boldsymbol{l}'_T}{(2\pi)^2}\int_0^1\mathrm{d}\alpha\,\frac{(\boldsymbol{l}'_T + \boldsymbol{k}_T)\cdot\boldsymbol{k}_T}{\boldsymbol{k}_T^2}\frac{1}{\left[\alpha(\boldsymbol{l}'_T + \boldsymbol{k}_T)^2 + (1-\alpha)(\boldsymbol{l}'^2_T + L^2)\right]^2} \\ &= \frac{1}{2P^+}\int_0^1\mathrm{d}\alpha\int\frac{\mathrm{d}^2\boldsymbol{l}'_T}{(2\pi)^2}\frac{(\boldsymbol{l}'_T + \boldsymbol{k}_T)\cdot\boldsymbol{k}_T}{\boldsymbol{k}_T^2}\frac{1}{\left[\boldsymbol{l}'^2_T + 2\alpha\,\boldsymbol{l}'_T\cdot\boldsymbol{k}_T + \alpha\,\boldsymbol{k}_T^2 + (1-\alpha)L^2\right]^2} \\ &\overset{l'_T\to\boldsymbol{l}_T - \alpha\boldsymbol{k}_T}{=} \frac{1}{2P^+}\int_0^1\mathrm{d}\alpha\int\frac{\mathrm{d}^2\boldsymbol{l}_T}{(2\pi)^2}\frac{1-\alpha}{\left[\boldsymbol{l}_T^2 + \alpha(1-\alpha)\boldsymbol{k}_T^2 + (1-\alpha)L^2\right]^2} \\ &= \frac{1}{8P^+\pi}\int_0^1\mathrm{d}\alpha\,\frac{1}{\alpha\,\boldsymbol{k}_T^2 + L^2} = \frac{1}{8P^+\pi}\frac{1}{\boldsymbol{k}_T^2}\ln\frac{\boldsymbol{k}_T^2 + L^2}{L^2}. \end{aligned} \tag{3.37}$$

Therefore, the result for the Sivers function is [31]

$$\begin{aligned} f_{1T}^{\perp}(x, k_T^2) &= \frac{e_c^2\,g_s^2}{4(2\pi)^4}\frac{M\,(m + xM)}{\boldsymbol{k}_T^2\,(k^2 - m^2)}\ln\frac{\boldsymbol{k}_T^2 + L^2}{L^2} \\ &= -\frac{e_c^2\,g_s^2}{4(2\pi)^4}\frac{M\,(1-x)\,(m + xM)}{\boldsymbol{k}_T^2\,(\boldsymbol{k}_T^2 + L^2)}\ln\frac{\boldsymbol{k}_T^2 + L^2}{L^2}. \end{aligned} \tag{3.38}$$

We can integrate the Sivers function over k_T^2 using a cutoff Λ^2 as we did for f_1 and obtain, e.g.,

$$f_{1T}^{\perp(1)}(x) = -\frac{e_c^2\,g_s^2\,(1-x)\,(m + xM)}{32(2\pi)^3 M}\ln\frac{\Lambda^2 + L^2}{L^2}. \tag{3.39}$$

Ex. 3.3

Calculate the Boer-Mulders function using the same procedure as for the Sivers function

3.4 Connection with impact-parameter space GPDs

The relation between the Sivers function and the impact-parameter representation of generalized parton distribution functions has been proposed for the first time in Ref. [40] and worked out in detail for the spedific case of the spectator model in Ref. [41].

The unpolarized distribution function can be written in terms of light-cone wave functions [41]

$$f_1(x, \boldsymbol{k}_T^2) = \frac{1}{4\pi}\Big(|\psi_+^+|^2 + |\psi_-^+|^2\Big) \tag{3.40}$$

where

$$\psi_+^+(x, \boldsymbol{k}_T) = (m + xM)\,\phi/x, \tag{3.41}$$
$$\psi_-^+(x, \boldsymbol{k}_T) = -(k^x + \mathrm{i}k^y)\,\phi/x, \tag{3.42}$$
$$\psi_+^-(x, \boldsymbol{k}_T) = (k^x - \mathrm{i}k^y)\,\phi/x, \tag{3.43}$$
$$\psi_-^-(x, \boldsymbol{k}_T) = (m + xM)\,\phi/x, \tag{3.44}$$
$$\phi(x, \boldsymbol{k}_T^2) = -\frac{g_s}{\sqrt{1-x}}\frac{x(1-x)}{\boldsymbol{k}_T^2 + L^2}. \tag{3.45}$$

The wave functions $\psi_\pm^\pm$ give the probability to find in a proton with positive/negative helicity (proton moving in the z direction) a quark with positive/negative helicity, longitudinal momentum fraction x and transverse momentum $\boldsymbol{k}_T$, together with a scalar diquark. The superscript refers to the proton helicity and the subscript to the quark helicity.

For a proton polarized in the y direction we obtain

$$\begin{aligned}
\psi_+^{\uparrow_y}(x, \boldsymbol{k}_T) &= \frac{1}{\sqrt{2}}(\psi_+^+ + \mathrm{i}\psi_+^-) = \frac{1}{\sqrt{2}}[(m + xM) + \mathrm{i}k^x + k^y]\phi/x,\\
\psi_-^{\uparrow_y}(x, \boldsymbol{k}_T) &= \frac{1}{\sqrt{2}}(\psi_-^+ + \mathrm{i}\psi_-^-) = \frac{1}{\sqrt{2}}[\mathrm{i}(m + xM) - k^x - \mathrm{i}k^y]\phi/x\\
\psi_+^{\downarrow_y}(x, \boldsymbol{k}_T) &= -\frac{1}{\sqrt{2}}(\psi_+^+ - \mathrm{i}\psi_+^-) = \frac{1}{\sqrt{2}}[(m + xM) + \mathrm{i}k^x + k^y]\phi/x,\\
\psi_-^{\downarrow_y}(x, \boldsymbol{k}_T) &= -\frac{1}{\sqrt{2}}(\psi_-^+ - \mathrm{i}\psi_-^-) = \frac{1}{\sqrt{2}}[\mathrm{i}(m + xM) - k^x - \mathrm{i}k^y]\phi/x
\end{aligned} \tag{3.46}$$

The impact-parameter-dependent (IPD) quark distribution for target polarization along the $\pm y$ direction and proton moving in the z direction reads [41]

$$f_{q/p^{\uparrow y}}(x, \boldsymbol{b}_T) = \mathcal{H}(x, \boldsymbol{b}_T) + \frac{1}{2M}\frac{\partial}{\partial b^x}\,\mathcal{E}(x, \boldsymbol{b}_T) \tag{3.47}$$
$$f_{q/p^{\downarrow y}}(x, \boldsymbol{b}_T) = \mathcal{H}(x, \boldsymbol{b}_T) - \frac{1}{2M}\frac{\partial}{\partial b^x}\,\mathcal{E}(x, \boldsymbol{b}_T). \tag{3.48}$$

$\frac{\partial}{\partial b^x}\,\mathcal{E} > 0$ corresponds to a preference of the quark to be on in the $+x$ side.

In the spectator model, the transverse momentum $\boldsymbol{k}_T$ is the Fourier conjugate to the distance between the quark and the spectator, $\boldsymbol{c}_T = \boldsymbol{r}_{Tq} - \boldsymbol{r}_{T(2q)}$. However, there is a simple relation between

$\boldsymbol{c}_T$ and the impact parameter $\boldsymbol{b}_T$, which is defined as the distance of the quark to the transverse center of momentum $\boldsymbol{b}_T = \boldsymbol{r}_{Tq} - \boldsymbol{R}_T$, where

$$\boldsymbol{R}_T = x\boldsymbol{r}_{Tq} + (1-x)\boldsymbol{r}_{T(2q)}. \tag{3.49}$$

so that

$$\boldsymbol{b}_T = \boldsymbol{r}_{Tq} - \boldsymbol{R}_T = (1-x)\boldsymbol{c}_T. \tag{3.50}$$

Therefore, there is a simple relation between quark densities in impact-parameter space and in transverse-coordinate space

$$f_{q/p^{\uparrow y}}(x, \boldsymbol{b}_T) = \frac{1}{(1-x)^2}\, f_{q/p^{\uparrow y}}(x, \boldsymbol{c}_T)\Big|_{\boldsymbol{c}_T = \boldsymbol{b}_T/(1-x)} \tag{3.51}$$

Introducting the Fourier transforms

$$\tilde{\psi}(x, \boldsymbol{c}_T) = \int \frac{d^2\boldsymbol{k}_T}{(2\pi)^2}\, e^{i\boldsymbol{k}_T\cdot\boldsymbol{c}_T}\, \psi(x, k_T) \tag{3.52}$$

we can finally write

$$f_{q/p^{\uparrow y}}(x, \boldsymbol{b}_T) = \frac{1}{(1-x)^2}\,\frac{1}{4\pi}\Big(|\tilde{\psi}_+^{\uparrow y}(x, \boldsymbol{c}_T)|^2 + |\tilde{\psi}_-^{\uparrow y}(x, \boldsymbol{c}_T)|^2\Big)\Big|_{\boldsymbol{c}_T = \boldsymbol{b}_T/(1-x)} \tag{3.53}$$

and similarly for the ↓ state.

The Fourier transforms of the light-cone wave functions are [41]

$$\tilde{\psi}_+^{\uparrow y}(x, \boldsymbol{c}_T)|^2 = \frac{1}{\sqrt{2}}\Big[(m + xM) + \frac{\partial}{\partial c^x} - i\frac{\partial}{\partial c^y}\Big]\tilde{\phi}(x, \boldsymbol{c}_T)/x \tag{3.54}$$

$$\tilde{\psi}_-^{\uparrow y}(x, \boldsymbol{c}_T)|^2 = \frac{1}{\sqrt{2}}\Big[i(m + xM) + i\frac{\partial}{\partial c^x} - \frac{\partial}{\partial c^y}\Big]\tilde{\phi}(x, \boldsymbol{c}_T)/x \tag{3.55}$$

and similarly for the ↓ state, with

$$\begin{aligned}\tilde{\phi}(x, \boldsymbol{c}_T) &= \int \frac{d^2\boldsymbol{k}_T}{(2\pi)^2}\, e^{i\boldsymbol{k}_T\cdot\boldsymbol{c}_T}\, \phi(x, \boldsymbol{k}_T^2) \\ &= -\frac{g_s x(1-x)}{\sqrt{1-x}}\int \frac{d^2\boldsymbol{k}_T}{(2\pi)^2}\, e^{i\boldsymbol{k}_T\cdot\boldsymbol{c}_T}\,\frac{1}{\boldsymbol{k}_T^2 + L^2} = -\frac{g_s}{2\pi}x\sqrt{1-x}\,K_0(|c_T|\,L),\end{aligned} \tag{3.56}$$

where K_0 stands for the first modified Bessel function.

It turns out that

$$\begin{aligned}|\tilde{\psi}_+^{\uparrow y}(x, \boldsymbol{c}_T)|^2 + |\tilde{\psi}_-^{\uparrow y}(x, \boldsymbol{c}_T)|^2 &= \Big[(m + xM)\tilde{\phi}/x + \frac{\partial}{\partial c^x}\,\tilde{\phi}/x\Big]^2 + \Big[\frac{\partial}{\partial c^y}\,\tilde{\phi}/x\Big]^2 \\ &= \frac{1}{x^2}\Big[(m + xM)^2\tilde{\phi}^2 + \Big(\frac{\partial}{\partial c^x}\,\tilde{\phi}\Big)^2 + \Big(\frac{\partial}{\partial c^y}\,\tilde{\phi}\Big)^2 + 2(m + xM)\,\tilde{\phi}\,\frac{\partial}{\partial c^x}\,\tilde{\phi}\Big]\end{aligned} \tag{3.57}$$

The analogous calculation for the $\downarrow$ state would lead to the same result, except for a different sign in the last term. Comparing with Eq. (3.53) and Eq. (3.47) we can find an expression for the function $\mathcal{E}$ in the diquark model, namely [41]

$$\frac{1}{2M}\frac{\partial}{\partial b^x}\mathcal{E}(x,\boldsymbol{b}_T) = \frac{1}{4\pi}\frac{2}{x^2(1-x)^2}(m+xM)\,\tilde{\phi}\,\frac{\partial}{\partial c^x}\,\tilde{\phi} \tag{3.58}$$

The transverse-momentum-dependent quark distribution for an unpolarized quark in a polarized target reads [22]

$$\begin{aligned} f_{q/p^\uparrow}(x,k_T) &= f_1^q(x,k_T^2) - f_{1T}^{\perp q}(x,k_T^2)\,\frac{\epsilon^{\mu\nu\rho\sigma}P_\mu k_\nu S_\rho n_\sigma}{M\,(P\cdot n)} \\ &= f_1^q(x,k_T^2) - f_{1T}^{\perp q}(x,k_T^2)\,\frac{(\hat{\boldsymbol{P}}\times\boldsymbol{k}_T)\cdot\boldsymbol{S}}{M}, \end{aligned} \tag{3.59}$$

$f_{1T}^{\perp q} > 0$ corresponds to a preference of the quark to move to the left if the proton is moving towards the observer and the proton spin is pointing upwards.

The average transverse momentum k_T^α/M is

$$\langle k_T^\alpha/M\rangle = \int \mathrm{d}^2\boldsymbol{k}_T\,\frac{k_T^\alpha}{M}\,f_{q/p^\uparrow}(x,k_T) = \epsilon_T^{\alpha\sigma}S_{T\sigma}\,f_{1T}^{\perp(1)} \tag{3.60}$$

Using expressions (3.30) and (3.36) we can write

$$\begin{aligned} \langle k_T^\alpha/M\rangle &= -\epsilon_T^{\alpha\sigma}S_{T\sigma}\,\frac{e_c^2\,g_s^2}{2M(2\pi)^3}\,(1-x)\,(xM+m) \\ &\times\int \mathrm{d}^2\boldsymbol{k}_T\int\frac{\mathrm{d}^2\boldsymbol{l}_T}{(2\pi)^2}\,\frac{(\boldsymbol{k}_T-\boldsymbol{l}_T)\cdot\boldsymbol{k}_T}{[\boldsymbol{k}_T^2+L^2]\,[(\boldsymbol{k}_T-\boldsymbol{l}_T)^2+m_g^2][\boldsymbol{l}_T^2+L^2]} \end{aligned} \tag{3.61}$$

Introducing

$$\Gamma^\rho\Big((\boldsymbol{k}_T-\boldsymbol{l}_T)^2\Big) = -\mathrm{i}\frac{e_c^2}{2}\,\frac{(k_T-l_T)^\rho}{(\boldsymbol{k}_T-\boldsymbol{l}_T)^2+m_g^2}, \tag{3.62}$$

we can write

$$\langle k_T^\alpha/M\rangle = -\epsilon_T^{\alpha\sigma}S_{T\sigma}\,\frac{(xM+m)}{2\pi\,x^2}\int\frac{\mathrm{d}^2\boldsymbol{k}_T}{(2\pi)^2}\int\frac{\mathrm{d}^2\boldsymbol{l}_T}{(2\pi)^2}\,\frac{k_{T\rho}}{M}\,\phi(x,\boldsymbol{k}_T^2)\,\phi(x,\boldsymbol{l}_T^2)\,\mathrm{i}\Gamma^\rho\Big((\boldsymbol{k}_T-\boldsymbol{l}_T)^2\Big), \tag{3.63}$$

where we can see that a convolution of three transverse-momentum dependent terms appears. It is convenient now to Fourier-transform this expression, so that it becomes a simple product in position space, since in general

$$\int\frac{\mathrm{d}^2\boldsymbol{k}_T}{(2\pi)^2}\int\frac{\mathrm{d}^2\boldsymbol{l}_T}{(2\pi)^2}\,f(\boldsymbol{k}_T)\,K(\boldsymbol{k}_T-\boldsymbol{l}_T)\,g(\boldsymbol{l}_T) = \int \mathrm{d}^2\boldsymbol{c}_T\,\tilde{f}(\boldsymbol{c}_T)\,\tilde{K}(\boldsymbol{c}_T)\,\tilde{g}(\boldsymbol{c}_T). \tag{3.64}$$

It follows that

$$\begin{aligned}\langle k_T^\alpha/M\rangle &= -\epsilon_T^{\alpha\sigma} S_{T\sigma}\,\frac{(xM+m)}{2\pi\,x^2}\int \mathrm{d}^2\boldsymbol{c}_T\,\tilde{\phi}(x,\boldsymbol{c}_T)\,\frac{\partial}{\partial c^\rho}\tilde{\phi}(x,\boldsymbol{c}_T)\,\tilde{\Gamma}^\rho(\boldsymbol{c}_T)\\ &= -\epsilon_T^{\alpha\sigma} S_{T\sigma}\int \mathrm{d}^2\boldsymbol{b}_T\,\frac{1}{2M}\,\frac{\partial}{\partial b^\rho}\,\mathcal{E}(x,\boldsymbol{b}_T)\,\tilde{\Gamma}^\rho(\boldsymbol{b}_T/(1-x)).\end{aligned} \tag{3.65}$$

We established a connection between the function $\mathcal{E}$ and the (1) transverse moment of the Sivers function. The term $\tilde{\Gamma}^\rho$ represents the final-state interaction that is essential to transfer an asymmetry in transverse space to an asymmetry in transverse momentum. Note that in this case we can compute also $\tilde{\Gamma}^\rho$

$$\tilde{\Gamma}^\rho(\boldsymbol{c}_T) = \int \frac{\mathrm{d}^2\boldsymbol{k}_T}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_T\cdot\boldsymbol{c}_T}\,\Gamma^\rho(\boldsymbol{k}_T) = -\mathrm{i}\frac{e_c^2}{2}\int \frac{\mathrm{d}^2\boldsymbol{k}_T}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_T\cdot\boldsymbol{c}_T}\,\frac{l_T^\rho}{\boldsymbol{l}_T^2+m_g^2} \overset{m_g\to 0}{=} \frac{e_c^2}{4\pi}\,\frac{c_T^\rho}{\boldsymbol{c}_T^2} \tag{3.66}$$

4

The gauge link

This chapter is based on Refs. [33, 79]

So far we used the following definition for the correlation function

$$\Phi_{ij}(p, P, S) = \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ip\cdot\xi} \langle P, S \,|\, \bar{\psi}_j(0)\, \psi_i(\xi) \,|\, P, S \rangle \tag{4.1}$$

or alternatively

$$\Phi_{ij}(x, p_T) = \int \frac{d\xi^- \, d^2\xi_T}{(2\pi)^3} \, e^{ip\cdot\xi} \, \langle P|\bar{\psi}_j(0)\, \psi_i(\xi)|P\rangle \Big|_{\xi^+=0} \tag{4.2}$$

with $p^+ = xP^+$.

It turns out that something is missing. The reason can be easily understood: the correlator as defined above is not gauge invariant, because the two quark field operators are at two different positions. If we perform a local (Abelian for now) gauge transformation on the fields

$$\psi(\xi) \to e^{i\alpha(\xi)}\, \psi(\xi) \tag{4.3}$$

the correlator evidently changes. This is something to worry about because the parton distribution functions composing the correlator can be directly measured and they should be gauge invariant.

To fix the problem, we have to insert a gauge link or Wilson line in between the quark fields, with the following gauge transformation properties

$$\mathcal{U}(\xi_1, \xi_2) \to e^{i\alpha(\xi_1)}\, \mathcal{U}(\xi_1, \xi_2)\, e^{-i\alpha(\xi_2)}. \tag{4.4}$$

If we consider only the leading twist contributions, it turns out that the proper gauge invariant definition of the quark-quark correlator is

$$\Phi_{ij}(x, p_T) = \int \frac{d\xi^- \, d^2\xi_T}{(2\pi)^3} \, e^{ip\cdot\xi} \, \langle P|\bar{\psi}_j(0) \, \mathcal{U}^{n_-}_{(0,+\infty)} \, \mathcal{U}^{n_-}_{(+\infty,\xi)} \, \psi_i(\xi)|P\rangle\Big|_{\xi^+=0} \tag{4.5}$$

where the gauge links (Wilson lines) are defined as

$$\mathcal{U}^{n_-}_{(0,+\infty)} = \mathcal{U}^{n_-}(0^-, \infty^-; \mathbf{0}_T) \, \mathcal{U}^T(\mathbf{0}_T, \boldsymbol{\infty}_T; \infty^-), \tag{4.6}$$

$$\mathcal{U}^{n_-}_{(+\infty,\xi)} = \mathcal{U}^T(\boldsymbol{\infty}_T, \boldsymbol{\xi}_T; \infty^-) \, \mathcal{U}^{n_-}(\infty^-, \xi^-, \boldsymbol{\xi}_T). \tag{4.7}$$

Here $\mathcal{U}^{n_-}(a^-, b^-; \boldsymbol{c}_T)$ indicates a Wilson line running along the minus direction from $[a^-, 0, \boldsymbol{c}_T]$ to $[b^-, 0, \boldsymbol{c}_T]$, while $\mathcal{U}^T(\boldsymbol{a}_T, \boldsymbol{b}_T; c^-)$ indicates a Wilson line running in the transverse direction from $[c^-, 0, \boldsymbol{a}_T]$ to $[c^-, 0, \boldsymbol{b}_T]$, i.e.

$$\mathcal{U}^{n_-}(a^-, b^-; \boldsymbol{c}_T) = \mathcal{P}\exp\Big[-ig\int_{a^-}^{b^-} d\eta^- A^+(\eta^-, 0, \boldsymbol{c}_T)\Big], \tag{4.8}$$

$$\mathcal{U}^T(\boldsymbol{a}_T, \boldsymbol{b}_T; c^-) = \mathcal{P}\exp\Big[-ig\int_{\boldsymbol{a}_T}^{\boldsymbol{b}_T} d\boldsymbol{\eta}_T \cdot \boldsymbol{A}_T(c^-, 0, \boldsymbol{\eta}_T)\Big]. \tag{4.9}$$

In particular

$$\mathcal{U}^{n_-}(\infty^-, \xi^-, \boldsymbol{\xi}_T) = \mathcal{P}\exp\Big[-ig\int_{\infty^-}^{\xi^-} d\eta^- A^+(\eta^-, 0, \boldsymbol{\xi}_T)\Big] \approx 1 - ig\int_{\infty^-}^{\xi^-} d\eta^- A^+(\eta^-, 0, \boldsymbol{\xi}_T) \tag{4.10}$$

$$\mathcal{U}^T(\boldsymbol{\infty}_T, \boldsymbol{\xi}_T; \infty^-) = \mathcal{P}\exp\Big[-ig\int_{\boldsymbol{\infty}_T}^{\boldsymbol{\xi}_T} d\boldsymbol{\eta}_T \cdot \boldsymbol{A}_T(\infty^-, 0, \boldsymbol{\eta}_T)\Big] \approx 1 - ig\int_{\boldsymbol{\infty}_T}^{\boldsymbol{\xi}_T} d\boldsymbol{\eta}_T \cdot \boldsymbol{A}_T(\infty^-, 0, \boldsymbol{\eta}_T) \tag{4.11}$$

The correlator in Eq. (4.5) is the one appearing in semi-inclusive DIS. In different processes the structure of the gauge link can change [21, 35, 36, 47].

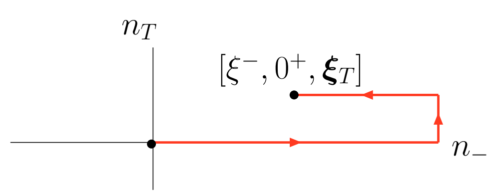


Figure 4.1. Gauge link for semi-inclusive DIS.

The gauge link can be derived by calculating the leading-twist contributions of diagrams of the type shown in Fig. 4.2 and their Hermitean conjugates. The presence of the transverse momentum dependence is essential, so we don't want to calculate simply inclusive DIS (where the transverse momentum is integrated over), but we suppose to do the calculation for jet DIS. The formula for the hadronic tensor can be inferred from Eq. (1.58) once we replace

$$\Delta(z, \boldsymbol{k}_T) \to \gamma^+ \, \delta(1 - z) \, \delta^{(2)}(\boldsymbol{k}_T). \tag{4.12}$$

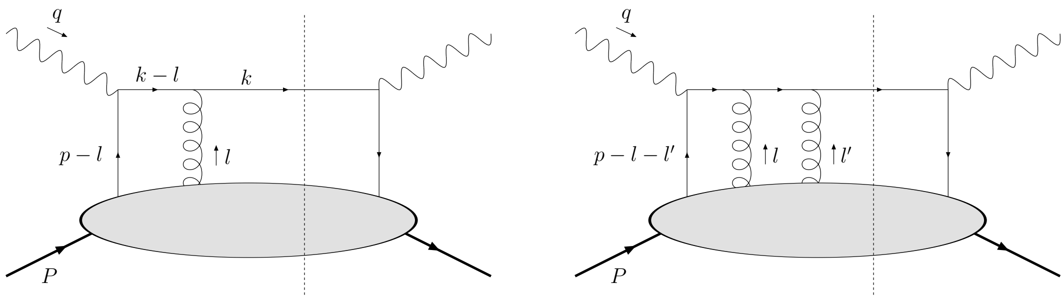


Figure 4.2. Examples of graphs contributing to the gauge link.

The formula for the hadronic tensor closely resembles the one we obtained for inclusive DIS, but now with the unintegrated correlation function, i.e.

$$2MW^{\mu\nu}(q,P,S,P_h) = 2\,\mathrm{Tr}\Big(\Phi(x_B,\boldsymbol{p}_{jT},S)\,\gamma^\mu\,\gamma^+\,\gamma^\nu\Big). \tag{4.13}$$

Let's take a look at the first diagram of Fig. 4.2 . We could write it as

$$2MW^{(a)}_{\mu\nu} \propto \int \mathrm{d}p^-\ \mathrm{d}^4 l\ \mathrm{Tr}\Big(\gamma_\alpha\,\frac{\not k - \not l + m}{(k-l)^2 - m^2 + \mathrm{i}\epsilon}\,\gamma_\nu \Phi^\alpha_A(p,p-l)\gamma_\mu\,(\not k + m)\Big)\Big|_{k=p+q} \tag{4.14}$$

where we introduced

$$\Phi^\alpha_{Aij}(p,p-l) = \int \frac{\mathrm{d}^4\xi}{(2\pi)^4}\,\frac{\mathrm{d}^4\eta}{(2\pi)^4}\ \mathrm{e}^{\mathrm{i}p\cdot\xi}\,\mathrm{e}^{\mathrm{i}l\cdot(\eta-\xi)}\langle P,S\,|\,\bar\psi_i(0)\,gA^\alpha(\eta)\,\psi_j(\xi)\,|\,P,S\,\rangle \tag{4.15}$$

so that

$$\begin{aligned} 2MW^{(a)}_{\mu\nu} \propto &\int \mathrm{d}p^-\ \mathrm{d}l^+\ \mathrm{d}^2\boldsymbol{l}_T \int \frac{\mathrm{d}^4\xi}{(2\pi)^4}\,\frac{\mathrm{d}\eta^-\ \mathrm{d}^2\boldsymbol{\eta}_T}{(2\pi)^3} e^{ip\cdot\xi} e^{i\,l\cdot(\eta-\xi)} \\ &\times \langle P,S\,|\bar\psi(0)\gamma_\mu\gamma^+\gamma_\alpha\,\frac{\not k - \not l + m}{(k-l)^2 - m^2 + i\epsilon}\,\gamma_\nu g A^\alpha(\eta)\psi(\xi)|P,S\,\rangle\Big|_{\eta^+=0}, \end{aligned} \tag{4.16}$$

where Φ^α_A is made explicit, the l^- integrations is performed. In the expression after the second equal sign, it is understood that $p^+ = x\,P^+$.

The quark propagator reads explicitly

$$\mathrm{i}\,\frac{\not k - \not l + m}{(k-l)^2 - m^2 + i\epsilon} \approx \mathrm{i}\,\frac{(\not k + m) - \gamma^-\,l^+ - \not l_T}{-2\,l^+\,k^- - (\boldsymbol{k}_T - \boldsymbol{l}_T)^2 - m^2 + i\epsilon}. \tag{4.17}$$

In the eikonal approximation, we took into consideration only the term $k^-\gamma^+$ in the numerator. Less obvious is the fact that there is another contribution, namely from the $\not l_T$ term, which turn out to be present only at $l^+ = 0$. Let's start first from the first kind of contribution. We approximate then the propagator with the standard eikonal propagator, see Eq. (3.12)

$$\mathrm{i}\,\frac{\not k - \not l + m}{(k-l)^2 - m^2 + i\epsilon} \approx \frac{\mathrm{i}}{2}\,\frac{\gamma^+}{-l^+ + \mathrm{i}\epsilon}. \tag{4.18}$$

We can use steps similar to what is described in Eq. (3.14) selecting γ_α to be only γ^- and consequently A^α to be A^+

$$\gamma^+\gamma_\alpha \frac{\not{k}-\not{l}+m}{(k-l)^2-m^2+i\epsilon}\gamma_\nu g A^\alpha(\eta) \approx \gamma^+ \frac{1}{2}\frac{\gamma^-\gamma^+}{-l^+ + \mathrm{i}\epsilon}\gamma_\nu g A^+(\eta) = -\gamma^+ \frac{gA^+(\eta)}{l^+ - \mathrm{i}\epsilon}\gamma_\nu \tag{4.19}$$

Then

$$\int \mathrm{d}l^+\,\mathrm{d}^2\boldsymbol{l}_T \frac{\mathrm{d}\eta^-\,\mathrm{d}^2\boldsymbol{\eta}_T}{(2\pi)^3} e^{i\,l\cdot(\eta-\xi)}\,\langle P,S\,|\overline{\psi}(0)\gamma_\mu\gamma^+\,\gamma_\nu\,\frac{gA^+(\eta)}{l^+ - i\epsilon}\psi(\xi)|P,S\,\rangle\Big|_{\eta^+=0} \tag{4.20}$$

and using

$$\int \mathrm{d}^2\boldsymbol{l}_T \frac{\mathrm{d}^2\boldsymbol{\eta}_T}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\,\boldsymbol{l}_T(\boldsymbol{\eta}_T-\boldsymbol{\xi}_T)} = \int \frac{\mathrm{d}^2\boldsymbol{\eta}_T}{(2\pi)^2}\,(2\pi)^2\,\delta^2(\boldsymbol{\eta}_T-\boldsymbol{\xi}_T) \tag{4.21}$$

$$\int \mathrm{d}l^+ \frac{\mathrm{e}^{\mathrm{i}\,l^+(\eta^- - \xi^-)}}{l^+ - i\epsilon}\,gA^+(\eta) = 2\pi\,\mathrm{i}\,gA^+(\eta)\,\theta(\eta^- - \xi^-) \tag{4.22}$$

we obtain

$$2MW^{(a)}_{\mu\nu} \propto \int \mathrm{d}p^- \int \frac{\mathrm{d}^4\xi}{(2\pi)^4}e^{ip\cdot\xi}\,\langle P,S\,|\overline{\psi}(0)\,\gamma_\mu\gamma^+\,\gamma_\nu\,(-ig)\int_{\infty^-}^{\xi^-}\mathrm{d}\eta^-\,A^+(\eta)\,\psi(\xi)|P,S\,\rangle\Big|_{\eta^+=0;\ \boldsymbol{\eta}_T=\boldsymbol{\xi}_T}\,. \tag{4.23}$$

By comparing this expression with Eq. (4.13), we can see that it corresponds to the $O(g)$ term in the expansion of the longitudinal part of the Wilson line $\mathcal{U}^-_{[\infty,\xi]}$ multiplying $\psi(\xi)$ in Eq. (4.5). The result of the diagram in Fig. 4.2b with two A^+-gluons gives the $O(g^2)$ term, etc. From the Hermitean conjugate diagram of Fig. 4.2a one obtains the $O(g)$ term in the expansion of the longitudinal part of the Wilson line $\mathcal{U}^-_{[0,\infty]}$ following $\bar{\psi(0)}$. Summing all these contributions we get

$$\Phi_{ij}(x,p_T) = \int \frac{\mathrm{d}\xi^-\,\mathrm{d}^2\xi_T}{(2\pi)^3}\,e^{\mathrm{i}p\cdot\xi}\,\langle P|\bar{\psi}_j(0)\,\mathcal{U}^{n_-}(0^-,\infty^-;\boldsymbol{0}_T)\,\mathcal{U}^{n_-}(\infty^-,\xi^-,\boldsymbol{\xi}_T)\,\psi_i(\xi)|P\rangle\Big|_{\xi^+=0} \tag{4.24}$$

If we choose at this point a light-cone gauge, where the A^+ vanish, the Wilson line can be reduced simply to unity. However, this is not yet the complete story. You can see it diagrammatically because we did not yet close the link path. You can also see it from the explicit calculation we did of the Sivers function. If in that case instead of using the gluon polarization sum in Feynman gauge we used the gluon polarization sum in light-cone gauge

$$d^{\mu\nu}(l;n_-) = -g^{\mu\nu} + \frac{l^\mu\,n_-^\nu + l^\nu\,n_-^\mu}{l\cdot n_-}, \tag{4.25}$$

our Sivers function would vanish as soon as we require the index μ to be $+$. There is something wrong, because the Sivers function cannot depend on the gauge.

In fact, there is now a last piece we have to worry about, namely transverse gluon fields at infinity. If we go back to the expression (4.17), we can check that naively all transverse gluon field contributions would be suppressed as $1/k^-$. However, if we consider only the transverse gluon field at ∞^-, i.e. $A_T[\infty^-, \eta^+, \boldsymbol{\eta}_T]$, this term does not depend anymore on η^-, so that we can easily perform the integration over this variable and obtain a $\delta(l^+)$ (soft gluons). From this term we get

$$\begin{aligned}\gamma^+ \not{l}_T \frac{\not{k} - \not{l} + m}{(k-l)^2 - m^2 + i\epsilon}\, \delta(l^+) &= \gamma^+ \not{l} \frac{\not{k} - \not{l} + m}{(k-l)^2 - m^2 + i\epsilon}\, \delta(l^+) \\ &= \gamma^+ (\not{l} - (\not{k} - m)) \frac{\not{k} - \not{l} + m}{(k-l)^2 - m^2 + i\epsilon}\, \delta(l^+) = -\gamma^+ \delta(l^+)\end{aligned} \tag{4.26}$$

We could do the first step because $l^- \gamma^+$ gives zero due to the presence of the initial γ^+, and the second step because we have an outgoing quark and the Dirac equation guarantees that $\bar{u}(k)(\not{k} - m) = 0$.

Finally, the integration over l^+, $\boldsymbol{l}_T$ and η_T gives

$$\int \mathrm{d}p^- \int \frac{\mathrm{d}\xi}{(2\pi)^4} e^{ip\cdot\xi}\, \langle P, S|\overline{\psi}(0)\, \gamma_\mu \gamma^+ \gamma_\nu\, (-ig) \int_{\infty_T}^{\xi_T} \mathrm{d}\eta_T \cdot A_T(\infty, \eta^+, \eta_T)\, \psi(\xi)|P, S\rangle \Big|_{\eta^+=0;\ \eta_T=\xi_T}, \tag{4.27}$$

which gives precisely the first term of the transverse link that is needed to modify Eq. (4.24) into the fully color gauge invariant correlation function of Eq. (4.5).

The final result is that the quark-quark correlator is now correctly defined with the required gauge link, at least at leading twist. We can now choose different gauges, but the final result should be unchanged. In particular, if we choose a light-cone gauge where $A^+ = 0$, then the longitudinal part of the gauge link is reduced to unity, but the transverse part in general is not. When using Feynman gauge, on the other hand, the transverse gauge link does not give a contribution, but the longitudinal does (this is exactly what happened in the spectator model calculation). Finally, we could choose a very specific gauge where $A^+ = 0$ and at the same time $A_T(+\infty^-) = 0$. In this case, we would get no contributions from the gauge link. We should however still be able to obtain a nonzero Sivers function in our spectator model, since this is a gauge invariant result. It would come from diagrams that we discarded because they could not contain an imaginary part *in normal gauges*, but they turn out to contain an imaginary part in this specific gauge.

4.1 Universality

Does the gauge link have the same form in different processes? If not, we should worry because we could not plug the quark-quark correlator we used so far into some different scattering process and we would run the risk of having to deal with *different* parton distribution functions in *different* processes.

It turns out that indeed the gauge link and the quark-quark correlator change depending on the process, but the nice news is that we can calculate exactly *how* it changes, and keep using universal distribution and fragmentation functions in different processes, once we correctly take into account these changes.

The easiest example is to analyze what happens in Drell-Yan scattering. The crucial difference is that instead of having an outgoing quark we have an incoming antiquark and that implies a small change in the form of the eikonal propagator. Let's take a look again at the eikonal approximation.

Let's work in Feynman gauge, so that we can use the eikonal approximation and avoid worrying about gluon fields at infinity. The relevant diagram is drawn in Fig. 4.3.

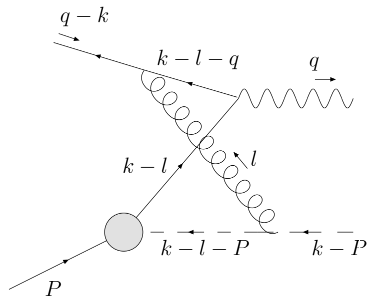


Figure 4.3.

$$\frac{\mathrm{i}(\not{k}-\not{q}-\not{l}+m)}{(k-q-l)^2-m^2+\mathrm{i}\epsilon} \approx \mathrm{i}\,\frac{-(q-k)^-\gamma^+}{2l^+(q-k)^-+\mathrm{i}\epsilon} = \frac{\mathrm{i}}{2}\,\frac{\gamma^+}{-l^+-\mathrm{i}\epsilon} \tag{4.28}$$

The only crucial difference from the steps in Eq. (3.12) is that here $(q-k)^- \geq 0$, because we have an incoming quark with momentum $q-k$ in the intial state. This leads to a different sign in the $\mathrm{i}\epsilon$. However, this makes a big difference for the calculation of the Sivers function in this case, since instead of using

$$\frac{1}{[l^+-\mathrm{i}\epsilon]} \to 2\pi\mathrm{i}\,\delta[l^+] \tag{4.29}$$

we now need

$$\frac{1}{[l^++\mathrm{i}\epsilon]} \to -2\pi\mathrm{i}\,\delta[l^+] \tag{4.30}$$

The final result would just have the opposite sign. More in general, in the construction of the gauge link we would have a gauge link running to $-\infty$ instead of $+\infty$, sometimes referred to as a past-pointing Wilson line instead of a future pointing one. The leading part of the transverse gauge link would also be at $-\infty^-$. The correlator would look like

$$\Phi_{ij}(x,p_T) = \int \frac{d\xi^-\,d^2\xi_T}{(2\pi)^3}\,e^{\mathrm{i}p\cdot\xi}\,\langle P|\bar{\psi}_j(0)\,\mathcal{U}^{n_-}_{(0,-\infty)}\,\mathcal{U}^{n_-}_{(-\infty,\xi)}\,\psi_i(\xi)|P\rangle\Big|_{\xi^+=0} \tag{4.31}$$

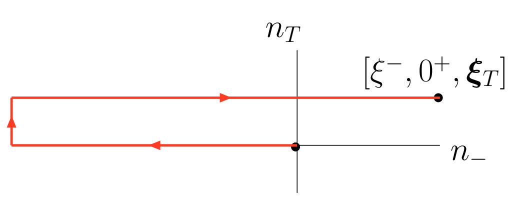


Figure 4.4.

What about fragmentation functions? The situation is more complex in that case because imaginary parts are present also in diagrams that are not related to the gauge link (at one loop, self energy and vertex corrections), and they don't change in different processes. Moreover, it turns out that in the "box diagram" the soft-gluon pole never gives a contribution at leading twist. As a consequence, there is no difference in sign between, for instance, fragmentation functions in semi-inclusive DIS and e^+e^- annihilation [48,75]. To prove this, it is sufficient to take a look at the

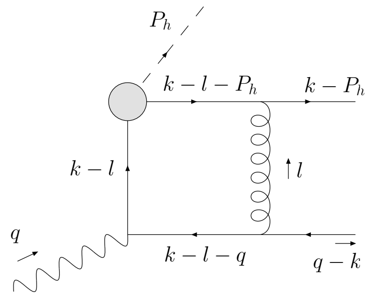


Figure 4.5.

analytic structure of the denominators of a typical box diagram (Fig. 4.5), similar to the integral occurring in Eq. (3.22). Let's reproduce a simplified version of that integral here

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\dots l_{T\rho}}{[l^2 - m_g^2 + \mathrm{i}\epsilon]\,[l^+ \pm \mathrm{i}\epsilon]\,[(k-l)^2 - m^2 + \mathrm{i}\epsilon][(k-l-P)^2 - M_s^2 + \mathrm{i}\epsilon]} \tag{4.32}$$

As we discussed already in the previous chapter, we need the real part of this integral, therefore we need to pick up two poles.

$$\frac{1}{[l^+ - \mathrm{i}\epsilon]} \to 2\pi\mathrm{i}\,\delta[l^+], \qquad \frac{1}{[(k-l-P)^2 - M_s^2 + \mathrm{i}\epsilon]} \to -2\pi\mathrm{i}\,\delta[(k-l-P)^2 - M_s^2], \tag{4.33}$$

$$\frac{1}{[l^2 - m_g^2 + \mathrm{i}\epsilon]} \to -2\pi\mathrm{i}\,\delta[l^2 - m_g^2], \qquad \frac{1}{[(k-l)^2 - m^2 + \mathrm{i}\epsilon]} \to -2\pi\mathrm{i}\,\delta[(k-l)^2 - m^2]. \tag{4.34}$$

In the calculation of a T-odd distribution function, we checked that it was possible to use only the combination of the first two δ functions.

In the case of the fragmentation function, the kinematical conditions change. The integral would look like

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\dots l_{T\rho}}{[l^2 - m_g^2 + \mathrm{i}\epsilon]\,[l^- \pm \mathrm{i}\epsilon]\,[(k-l)^2 - m^2 + \mathrm{i}\epsilon][(k-l-P)^2 - M_s^2 + \mathrm{i}\epsilon]} \tag{4.35}$$

In particular, if we choose the pick up the eikonal pole

$$\frac{1}{[l^- \pm \mathrm{i}\epsilon]} \to \mp 2\pi\mathrm{i}\,\delta[l^-] \tag{4.36}$$

to perform the l^- integration, we would be left with

$$\mp\mathrm{i} \int \frac{\mathrm{d}^2 \boldsymbol{l}_T\ \mathrm{d}l^-}{(2\pi)^3} \frac{\dots l_{T\rho}}{4k^-(k-P)^-\,[-\boldsymbol{l}_T^2 - m_g^2 + \mathrm{i}\epsilon]\,[l^+ - a - \frac{\mathrm{i}\epsilon}{(k-P)^-}]\,[l^+ - b - \frac{\mathrm{i}\epsilon}{k^-}]} \tag{4.37}$$

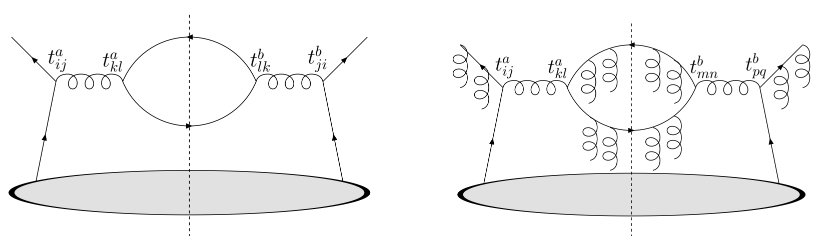
The $\boldsymbol{l}_T^2$ is out of question because otherwise we would have a vanishing numerator. $(k-P)^-$ must be positive because its the momentum of the on-shell spectator, implying that also k^- must be positive (P^- itself is positive). Therefore, if we try to perform the l^+ integration, we are forced to take *two* poles (e^+e^- annihilation) or none (SIDIS). In both cases, it turns out that the final result vanishes.

This shows that the eikonal pole cannot give a contribution. The other combination that works in both SIDIS and e^+e^- is to pick up the l^2 and $(k-l)^2$ poles. There is no change of sign in this particular contribution, thus the conclusion that fragmentation functions are universal, in an "easier" way that distribution functions.

4.2 More complex processes

We have seen shortly what happens in Drell-Yan compared to SIDIS. They are relatively simple because they have a colored state that can interact with the gluons only either in the initial (Drell-Yan) or final state (SIDIS). However, in hadron-hadron collisions there are colored states both in initial and final state and the situation becomes more complex. In particular, we have to take into account the non-Abelian nature of the color interaction.

As an example, let's consider one of the many partonic processes contributing to jet-jet production in proton-proton collisions, namely $q\bar{q} \to q'\bar{q}'$. The Feynman diagram at Born level is depicted in Fig. 4.6 (a).



(a) (b)

Figure 4.6.

The calculation of the tree-level diagram is done in detail e.g. in Peskin-Schroeder, at the end of Sec. 17.4. It corresponds, e.g., to $e^-e^+ \to \mu^-\mu^+$, the only difference being the color structure. The color matrices give, including color averaging

$$\begin{aligned}\frac{1}{N_c}\frac{1}{N_c}\, t^a_{il}\, t^b_{li}\, t^a_{jm}\, t^b_{mj} &= \frac{1}{N_c^2}\,\mathrm{Tr}[t^a t^b]\,\mathrm{Tr}[t^a t^b] = \frac{1}{N_c^2}\, T_F^2 \delta^{ab}\,\delta^{ab} \\ &= \frac{T_F^2}{N_c^2}\,\delta^{aa} = \frac{T_F^2}{N_c^2}\,(N_c^2-1) = \frac{T_F}{N_c}\,C_F = \frac{2}{9}.\end{aligned} \tag{4.38}$$

and the partonic cross-section reads

$$d\hat{\sigma}_{q\bar{q}\to q'\bar{q}'} = \frac{2\pi\alpha_S^2}{\hat{s}^2}\,\frac{2}{9}\left(\frac{\hat{t}^2}{\hat{s}^2} + \frac{\hat{u}^2}{\hat{s}^2}\right). \tag{4.39}$$

The inclusion of the quark-quark correlator without gauge link makes no difference, because the color indices of the two quark fields are forced to be the same by the color structure of the partonic scattering. In fact, using the relation

$$t^a_{ik}\, t^a_{mn} = T_F\Big(\delta_{in}\,\delta_{mk} - \frac{1}{N_c}\,\delta_{ik}\,\delta_{mn}\Big). \tag{4.40}$$

we can check that

$$\begin{aligned}
\Phi_{qi}\, t^a_{ik}\, t^b_{kq}\, t^a_{jl}\, t^b_{lj} &= T_F^2\, \Phi_{qi}\Big(\delta_{il}\,\delta_{kj} - \frac{1}{N_c}\,\delta_{ik}\,\delta_{jl}\Big)\Big(\delta_{kj}\,\delta_{ql} - \frac{1}{N_c}\,\delta_{kq}\,\delta_{lj}\Big)\\
&= T_F^2\, \Phi_{qi}\Big(\delta_{iq}\,\delta_{jj} + \frac{1}{N_c^2}\,\delta_{iq}\,\delta_{jj} - \frac{1}{N_c}\,\delta_{iq} - \frac{1}{N_c}\,\delta_{iq}\Big)\\
&= T_F^2\, \Phi_{qi}\,\delta_{iq}\,\frac{N_c^2-1}{N_c} = \Phi_{ii}\, T_F\, C_F
\end{aligned} \tag{4.41}$$

Including gauge links corresponds to including all possible gluon interactions drawn in Fig. 4.6 (b). We already know that the Dirac structure of the interaction is not a problem because it simplifies in the eikonal approximation and can be "pulled through" the Born-level quark-gluon vertices to form the gauge link. What is new in this case is the color structure. We are going to look in detail at what happens to it. To do this, we only have to indicate explicitly the color indices of the gauge link and of the quark-quark correlator without gauge link, which we'll denote as a $\tilde{\Phi}$. It is furthermore convenient to introduce the shorthand notation

$$U^{[+]} = \mathcal{U}^{n_-}_{(0,+\infty)}\,\mathcal{U}^{n_-}_{(+\infty,\xi)}, \qquad\qquad U^{[-]} = \mathcal{U}^{n_-}_{(0,-\infty)}\,\mathcal{U}^{n_-}_{(-\infty,\xi)}. \tag{4.42}$$

The color structure of the diagram with the correlator becomes [36, Eq. (21)]

$$\Big[\tilde{\Phi}_{qi}\, t^a_{ij}\, U^{[-]}_{jp}\, t^b_{pq}\Big]\Big[t^a_{kl}\, U^{[+]}_{lm}\, t^b_{mn}\, U^{[+]\dagger}_{nk}\Big] \tag{4.43}$$

The goal is to write the above expression in a way similar to the Born-level expression, Eq. (4.41), putting all extra structures inside Φ, now with its proper gauge link. Let's see in detail how this can be done:

$$\begin{aligned}
&T_F^2\, \tilde{\Phi}_{qi}\, U^{[-]}_{jp}\, U^{[+]}_{lm}\, U^{[+]\dagger}_{nk}\Big(\delta_{il}\,\delta_{jk}\,\delta_{pn}\,\delta_{qm} + \frac{1}{N_c^2}\,\delta_{ij}\,\delta_{kl}\,\delta_{pq}\,\delta_{mn} - \frac{1}{N_c}\,\delta_{ij}\,\delta_{kl}\,\delta_{pn}\,\delta_{qm} - \frac{1}{N_c}\,\delta_{il}\,\delta_{jk}\,\delta_{pq}\,\delta_{mn}\Big)\\
&= T_F^2\, \tilde{\Phi}_{qi}\Big(U^{[-]}_{jp}\, U^{[+]}_{iq}\, U^{[+]\dagger}_{pj} + \frac{1}{N_c^2}\, U^{[-]}_{iq}\, U^{[+]}_{lm}\, U^{[+]\dagger}_{ml} - \frac{1}{N_c}\, U^{[-]}_{ip}\, U^{[+]}_{lq}\, U^{[+]\dagger}_{pl} - \frac{1}{N_c}\, U^{[-]}_{jq}\, U^{[+]}_{im}\, U^{[+]\dagger}_{mj}\Big).
\end{aligned} \tag{4.44}$$

We note that

$$U^{[+]}_{im}\, U^{[+]\dagger}_{mj} = \delta_{ij} \tag{4.45}$$

and we remember that $\delta_{ii} = N_c$, to obtain

$$\begin{aligned}
&T_F^2\, \tilde{\Phi}_{qi}\Big(U^{[+]}_{iq}\,\mathrm{Tr}[U^{[-]}U^{[+]\dagger}] - \frac{1}{N_c}\, U^{[-]}_{iq}\Big)\\
&= T_F\, C_F\, \frac{N_c}{N_c^2-1}\, \tilde{\Phi}_{qi}\Big(U^{[+]}_{iq}\,\mathrm{Tr}[U^{[-]}U^{[+]\dagger}] - \frac{1}{N_c}\, U^{[-]}_{iq}\Big)
\end{aligned} \tag{4.46}$$

Therefore, the expression is equivalent to Eq. (4.41) once we identify

$$\Phi_{ii} = \tilde{\Phi}_{qi} \frac{1}{N_c^2 - 1} \left(N_c^2 \, U_{iq}^{[+]} \frac{\mathrm{Tr}[U^{[-]} U^{[+]\dagger}]}{N_c} - U_{iq}^{[-]} \right) \tag{4.47}$$

The new gauge link is not simply a $U^{[+]}$ as in SIDIS, nor a $U^{[-]}$ as in Drell-Yan, but rather a combination of the two, due to the fact that we have both incoming and outgoing colored states which are subject to gluon rescattering.

4.3 Consequences of different gauge links

Difference gauge links have the effect of modifying the T-odd distribution functions, which are sensitive to the presence of the gauge link. In particular, if the Sivers and Boer-Mulders function contribute in DIS with a certain sign, they will contribute with the opposite sign in Drell-Yan. In proton-proton collisions, for each different partonic diagram they acquire a different numerical prefactor, which is sometimes called a gluonic-pole strenght.

In the expression for the cross section, they have to be convoluted not with standard partonic cross-section, but rather with gluonic-pole cross section, which take into account the numerical factors we discussed above.

As a consequence of the gauge link, the T-odd distribution functions are multiplied by a (sub)process dependent factor $C_G^{[\mathcal{U}]}$. which is called a gluonic pole strength. The future pointing Wilson line in SIDIS leads to C_G^{SIDIS}=+1, while the past pointing Wilson line in DY gives the gluonic pole strength C_G^{DY}=−1. From this observation follows the important conclusion that the Sivers effect appears with opposite signs in SIDIS and DY [39,47]:

$$\text{Sivers effect in SIDIS:} \qquad \mathrm{d}\sigma_{\ell H \to \ell h X} \;\sim\; +f_{1T}^{\perp(1)}(x)\, \mathrm{d}\hat{\sigma}_{\ell q \to \ell q}\, D_1(z)\,, \tag{4.48a}$$

$$\text{Sivers effect in DY:} \qquad \mathrm{d}\sigma_{HH' \to \ell \bar{\ell} X} \;\sim\; -f_{1T}^{\perp(1)}(x)\, \bar{f}_1(x')\, \mathrm{d}\hat{\sigma}_{q\bar{q} \to \ell \bar{\ell}}\,. \tag{4.48b}$$

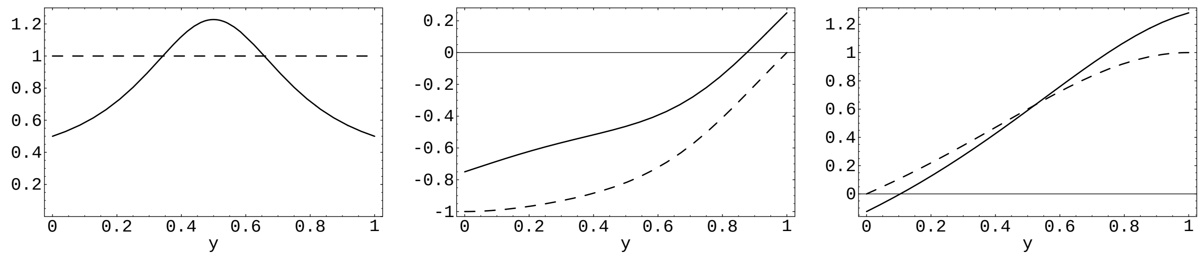
In partonic processes with more complex gauge links, the gluonic pole strength is not limited to just a sign [21,34,35]. The recipe consists in taking the result for the gauge-invariant correlation function for each partonic diagram, e.g. (4.47) and replace each gauge link structure with a certain number, namely

$$U^{[+]} \to +1, \qquad U^{[-]} \to -1, \qquad \mathrm{Tr}[U^{[-]} U^{[+]\dagger}] = \mathrm{Tr}[U^{[-]\dagger} U^{[+]}] \to N_c, \tag{4.49}$$

$$U^{[+]\dagger} \to -1, \qquad U^{[-]\dagger} \to +1, \qquad \mathrm{Tr}[U^{[+]} U^{[-]\dagger} U^{[+]}] \to +3, \tag{4.50}$$

so that the gluonic pole strenght for the $q\bar{q} \to q'\bar{q}'$ diagram is $C_G = N_c^2 + 1/N_c^2 - 1 = 5/4$.

It becomes particularly interesting when one considers a process where there are different Feynman diagrams that contribute to the hard process. Since the gauge link depends on the color-flow of the hard diagram the gluonic pole strength can in general be different for each contribution. For example, the gluonic pole strengths of the t and u channel contributions to identical quark scattering are C_G=$(N^2{-}5)/(N^2{-}1)$=$\frac{1}{2}$, whereas those of the tu interference diagrams are



$d\hat{\sigma}_{[q]q\to qq}/d\hat{\sigma}_{qq\to qq}$ $d\hat{\sigma}_{[q]\bar{q}\to q\bar{q}}/d\hat{\sigma}_{q\bar{q}\to q\bar{q}}$ $d\hat{\sigma}_{[q]g\to qg}/d\hat{\sigma}_{qg\to qg}$

Figure 4.7. Some examples of ratios of gluonic pole cross sections and partonic cross sections as functions of the variable $y\equiv-\hat{t}/\hat{s}$ for N=3 (solid line) and $N\to\infty$ (dashed line).

$C_G = -(N^2+3)/(N^2-1) = -\frac{3}{2}$. Therefore, the Sivers-effect contribution of identical quark scattering is seen to contain the gluonic pole matrix element $f_{1T}^{\perp(1)}(x)$ in the combination

$$f_{1T}^{\perp(1)}(x)\Big(\{\tfrac{1}{2}\} \quad + \{\tfrac{1}{2}\} \quad - \{-\tfrac{3}{2}\} \quad - \{-\tfrac{3}{2}\} \quad \Big), \tag{4.51}$$

rather than with the partonic cross section, which does not contain the weight factors between braces $\{\cdot\}$. This observation generalizes to all partonic processes and hence the quark-Sivers function $f_{1T}^{\perp(1)}$ is seen to appear with the hard functions

$$d\hat{\sigma}_{[q]a\to bc} = \sum_D C_G^{[\mathcal{U}(D)]}\, d\hat{\sigma}^{[D]}_{qa\to bc}\,, \tag{4.52}$$

where $d\hat{\sigma}^{[D]}$ is the squared amplitude expression of the Feynman diagram D and the summation runs over all diagrams D that can contribute to the process $qa\to bc$. The hard functions (4.52) are the *gluonic pole cross sections*. Hence, the gluonic pole cross sections are gauge-invariant weighted sums of Feynman diagrams that are, in general, distinct from the partonic cross sections that enter in spin-averaged processes (see Figure 4.7). The bracketed subscript $[q]$ indicates that in this example it is quark q that contributes the gluonic pole.

5

Semi-inclusive DIS at subleading twist

This chapter is based essentially on Ref. 23.

There are three crucial steps to go from leading twist to subleading twist. First of all, the study of the quark-quark correlation function has to be extended to subleading twist. Secondly, the contributions from quark-gluon-quark correlation functions has to be taken into consideration. Thirdly, it is important to keep track of the difference between assuming the proton and the outgoing hadron to be collinear (the frame where the distribution and fragmentation functions are defined) and the proton and photon to be collinear (the frame where the azimuthal angles appearing in the cross section are defined).

5.1 Quark-quark correlator up to subleading twist

Due to the presence of the gauge link, for the general parametrization of the correlator we have to take into account also the extra vector n_- [see Eq. (4.5)]. Because of this, new structures appear in the decomposition of Eq. (2.20) [57]. The full decomposition for a nucleon target has been studied in Ref. [58] and reads

$$
\begin{aligned}
\Phi(p, P, S|n_-) = {} & MA_1 + \not{P}A_2 + \not{p}A_3 + \frac{i}{2M}[\not{P}, \not{p}]\, A_4 + i(p\cdot S)\gamma_5\, A_5 + M\not{S}\gamma_5\, A_6 \\
& + \frac{p\cdot S}{M}\not{P}\gamma_5\, A_7 + \frac{p\cdot S}{M}\not{p}\gamma_5\, A_8 + \frac{[\not{P}, \not{S}]}{2}\gamma_5\, A_9 + \frac{[\not{p}, \not{S}]}{2}\gamma_5\, A_{10} \\
& + \frac{(p\cdot S)}{2M^2}[\not{P}, \not{p}]\gamma_5\, A_{11} + \frac{1}{M}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu P_\nu p_\rho S_\sigma\, A_{12} \\
& + \frac{M^2}{P\cdot n_-}\not{n}_-\, B_1 + \frac{iM}{2P\cdot n_-}[\not{P}, \not{n}_-]\, B_2 + \frac{iM}{2P\cdot n_-}[\not{p}, \not{n}_-]\, B_3 \\
& + \frac{1}{P\cdot n_-}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_5 P_\nu p_\rho n_{-\sigma}\, B_4 \\
& + \frac{1}{P\cdot n_-}\varepsilon^{\mu\nu\rho\sigma}P_\mu p_\nu n_{-\rho} S_\sigma\, B_5 + \frac{iM^2}{P\cdot n_-}(n_-\cdot S)\gamma_5\, B_6 \\
& + \frac{M}{P\cdot n_-}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu P_\nu n_{-\rho} S_\sigma\, B_7 + \frac{M}{P\cdot n_-}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu p_\nu n_{-\rho} S_\sigma\, B_8 \\
& + \frac{(p\cdot S)}{M(P\cdot n_-)}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu P_\nu p_\rho n_{-\sigma}\, B_9 + \frac{M(n_-\cdot S)}{(P\cdot n_-)^2}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu P_\nu p_\rho n_{-\sigma}\, B_{10} \\
& + \frac{M}{P\cdot n_-}(n_-\cdot S)\not{P}\gamma_5\, B_{11} + \frac{M}{P\cdot n_-}(n_-\cdot S)\not{p}\gamma_5\, B_{12} \\
& + \frac{M}{P\cdot n_-}(p\cdot S)\not{n}_-\gamma_5\, B_{13} + \frac{M^3}{(P\cdot n_-)^2}(n_-\cdot S)\not{n}_-\gamma_5\, B_{14} \\
& + \frac{M^2}{2P\cdot n_-}[\not{n}_-, \not{S}]\gamma_5\, B_{15} + \frac{(p\cdot S)}{2P\cdot n_-}[\not{P}, \not{n}_-]\gamma_5\, B_{16} + \frac{(p\cdot S)}{2P\cdot n_-}[\not{p}, \not{n}_-]\gamma_5\, B_{17} \\
& + \frac{(n_-\cdot S)}{2P\cdot n_-}[\not{P}, \not{p}]\gamma_5\, B_{18} + \frac{M^2(n_-\cdot S)}{2(P\cdot n_-)^2}[\not{P}, \not{n}_-]\gamma_5\, B_{19} + \frac{M^2(n_-\cdot S)}{2(P\cdot n_-)^2}[\not{p}, \not{n}_-]\gamma_5\, B_{20}\,,
\end{aligned}
\tag{5.1}
$$

where the B amplitudes are new compared to Eq. (2.20).

To simplify the discussion, let's first focus on the unpolarized part [24]

$$
\begin{aligned}
\Phi(p, P, S|n_-) = {} & MA_1 + \not{P}A_2 + \not{p}A_3 + \frac{i}{2M}[\not{P}, \not{p}]\, A_4 \\
& + \frac{M^2}{P\cdot n_-}\not{n}_-\, B_1 + \frac{iM}{2P\cdot n_-}[\not{P}, \not{n}_-]\, B_2 + \frac{iM}{2P\cdot n_-}[\not{p}, \not{n}_-]\, B_3 \\
& + \frac{1}{P\cdot n_-}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_5 P_\nu p_\rho n_{-\sigma}\, B_4
\end{aligned}
\tag{5.2}
$$

We keep only the leading and subleading terms in $1/P^+$

$$
\begin{aligned}
\Phi(P, p; n_-) \approx {} & P^+ (A_2 + xA_3)\, \not{n}_+ + P^+ \frac{\mathrm{i}}{2M}[\not{n}_+, \not{p}_T] A_4 + MA_1 + \not{p}_T A_3 \\
& + \left[\left(\frac{P^+ p^-}{M} - \frac{xM}{2}\right) A_4 + M(B_2 + xB_3)\right] \frac{\mathrm{i}}{2}[\not{n}_+, \not{n}_-] + \gamma_5 \epsilon_T^{\rho\sigma}\gamma_\rho p_{T\sigma} B_4
\end{aligned}
\tag{5.3}
$$

leading to

$$
\begin{aligned}
\Phi(x,p_T) &\equiv \int \mathrm{d}p^- \Phi^{[+]}(P,p;n_-) \\
&= \frac{1}{2}\left\{ f_1 \not{n}_+ + \mathrm{i} h_1^\perp \frac{[\not{p}_T, \not{n}_+]}{2M} \right\} + \frac{M}{2P^+}\left\{ e + f^\perp \frac{\not{p}_T}{M} + \mathrm{i} h \frac{[\not{n}_+, \not{n}_-]}{2} + g^\perp \gamma_5 \frac{\epsilon_T^{\rho\sigma}\gamma_\rho p_{T\sigma}}{M} \right\}.
\end{aligned}
\tag{5.4}
$$

Two crucial observations can be made at this point:

- were the B amplitudes set to zero, there would be a relation between the functions $h_1^\perp$ and h;
- were the B amplitudes set to zero, the function $g^\perp$ would not even exist.

The relations between the functions in the absence of the B amplitudes are known in the literature as "Lorentz invariance relations". Taking the example of the unpolarized case, the relation can be obtained as follows. First the relation between the functions and the amplitudes has to be sorted out, i.e.

$$
\begin{aligned}
f_1(x,p_T^2) &= 2P^+ \int \mathrm{d}p^-\,(A_2 + xA_3)\,, & h_1^\perp(x,p_T^2) &= 2P^+ \int \mathrm{d}p^-\,(-A_4)\,, \\
e(x,p_T^2) &= 2P^+ \int \mathrm{d}p^- A_1, & f^\perp(x,p_T^2) &= 2P^+ \int \mathrm{d}p^- A_3, \\
h(x,p_T^2) &= 2P^+ \int \mathrm{d}p^- \left(\frac{p\cdot P - xM^2}{M^2} A_4 + B_2 + xB_3 \right), & g^\perp(x,p_T^2) &= 2P^+ \int \mathrm{d}p^- B_4.
\end{aligned}
$$

Suppose we now set the B functions to zero, we obtain in particular

$$
h(x,p_T^2) = 2P^+ \int \mathrm{d}p^-\, \frac{p\cdot P - xM^2}{M^2}\, A_4. \tag{5.5}
$$

Introducing the symbols $\tau = p^2$ and $\sigma = 2p\cdot P$, which are related by [see e.g. Eq. 1.22]

$$
\tau = x\sigma - x^2M^2 - \boldsymbol{p}_T^2, \tag{5.6}
$$

we can then rewrite

$$
h_1^{\perp(1)}(x) = -\pi \int \mathrm{d}\boldsymbol{p}_T^2\, \mathrm{d}\sigma\, \mathrm{d}\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{p}_T^2)\, \frac{\boldsymbol{p}_T^2}{2M^2}\, A_4(\sigma,\tau) \tag{5.7}
$$

$$
h(x) = \pi \int \mathrm{d}\boldsymbol{p}_T^2\, \mathrm{d}\sigma\, \mathrm{d}\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{p}_T^2)\, \frac{\sigma - 2xM^2}{2M^2}\, A_4(\sigma,\tau). \tag{5.8}
$$

We observe that

$$
-\frac{\mathrm{d}}{\mathrm{d}x} h_1^{\perp(1)}(x) = \pi \int \mathrm{d}\boldsymbol{p}_T^2\, \mathrm{d}\sigma\, \mathrm{d}\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{p}_T^2)\, \frac{\boldsymbol{p}_T^2}{2M^2}\, \frac{\partial A_4(\sigma,\tau)}{\partial\tau}\, \frac{\mathrm{d}\tau}{\mathrm{d}x} \tag{5.9}
$$

since

$$
\frac{\partial A_4(\sigma,\tau)}{\partial\tau} = -\frac{\partial A_4(\sigma,\tau)}{\partial \boldsymbol{p}_T^2} \tag{5.10}
$$

we obtain

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}x}\, h_1^{\perp(1)}(x) &= -\pi \int \mathrm{d}\boldsymbol{p}_T^2\, \mathrm{d}\sigma\ \mathrm{d}\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{p}_T^2)\, \frac{\boldsymbol{p}_T^2}{2M^2}\, \frac{\partial A_4(\sigma,\tau)}{\partial \boldsymbol{p}_T^2}\, (\sigma - 2xM^2) \\
&= -\pi \int \mathrm{d}\sigma\ \mathrm{d}\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{p}_T^2) \frac{\boldsymbol{p}_T^2}{2M^2}\, A_4(\sigma,\tau)\Big|_0^\infty \\
&\quad + \pi \int \mathrm{d}\boldsymbol{p}_T^2\, \mathrm{d}\sigma\ \mathrm{d}\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{p}_T^2)\, \frac{\sigma - 2xM^2}{2M^2}\, A_4(\sigma,\tau),
\end{aligned} \tag{5.11}
$$

$$
-\frac{\mathrm{d}}{\mathrm{d}x}\, h_1^{\perp(1)}(x) = h(x) \tag{5.12}
$$

In the last step we assumed that the first term vanishes (otherwise p_T-dependent functions could never be integrated between 0 and $infty$).

With similar steps, one can obtain the following relations

$$
g_T(x) = g_1(x) + \frac{\mathrm{d}}{\mathrm{d}x}\, g_{1T}^{\perp}(x), \tag{5.13}
$$

$$
g_L^{\perp}(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\, g_T^{\perp(1)}(x), \tag{5.14}
$$

$$
h_L(x) = h_1(x) - \frac{\mathrm{d}}{\mathrm{d}x}\, h_{1L}^{\perp}(x), \tag{5.15}
$$

$$
h_T^{\perp}(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\, h_{1T}^{\perp(1)}(x), \tag{5.16}
$$

$$
h_{1L}^{\perp}(x, \boldsymbol{p}_T^2) = h_T(x, \boldsymbol{p}_T^2) - h_T^{\perp}(x, \boldsymbol{p}_T^2) \tag{5.17}
$$

$$
f_T(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\, f_{1T}^{\perp(1)}(x), \tag{5.18}
$$

$$
h(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\, h_1^{\perp(1)}(x), \tag{5.19}
$$

$$
e_L(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\, e_T^{(1)}(x), \tag{5.20}
$$

$$
f_{1T}^{\perp}(x, \boldsymbol{p}_T^2) = -f_L^{\perp}(x, \boldsymbol{p}_T^2). \tag{5.21}
$$

Ex. 5.1

Try to obtain the first relation using the parametrization of Eq. (2.20) *and check that it is violated when extending the parametrization as in Eq.* (5.1).

The full result for

$$
\begin{aligned}
\Phi(x, p_T) = \frac{1}{2}\Big\{ & f_1 \not{n}_+ - f_{1T}^{\perp}\, \frac{\epsilon_T^{\rho\sigma} p_{T\rho} S_{T\sigma}}{M}\, \not{n}_+ + g_{1s}\gamma_5 \not{n}_+ \\
& + h_{1T}\, \frac{[\not{S}_T, \not{n}_+]\gamma_5}{2} + h_{1s}^{\perp}\, \frac{[\not{p}_T, \not{n}_+]\gamma_5}{2M} + \mathrm{i}\, h_1^{\perp} \frac{[\not{p}_T, \not{n}_+]}{2M} \Big\} \\
+ \frac{M}{2P^+}\Big\{ & e - \mathrm{i}\, e_s\, \gamma_5 - e_T^{\perp}\, \frac{\epsilon_T^{\rho\sigma} p_{T\rho} S_{T\sigma}}{M}
\end{aligned}
$$

$$
\begin{aligned}
&+ f^{\perp}\,\frac{\not{p}_T}{M} - f_T'\,\epsilon_T^{\rho\sigma}\gamma_\rho S_{T\sigma} - f_s^{\perp}\,\frac{\epsilon_T^{\rho\sigma}\gamma_\rho p_{T\sigma}}{M}\\
&+ g_T'\,\gamma_5\not{S}_T + g_s^{\perp}\gamma_5\frac{\not{p}_T}{M} - g^{\perp}\gamma_5\,\frac{\epsilon_T^{\rho\sigma}\gamma_\rho p_{T\sigma}}{M}\\
&+ h_s\,\frac{[\not{n}_+,\not{n}_-]\gamma_5}{2} + h_T^{\perp}\,\frac{[\not{S}_T,\not{p}_T]\gamma_5}{2M} + \mathrm{i}\,h\frac{[\not{n}_+,\not{n}_-]}{2}\Big\}. \qquad (5.22)
\end{aligned}
$$

The distribution functions on the r.h.s. depend on x and p_T^2, except for the functions with subscript s, where we use the shorthand notation [76]

$$
g_{1s}(x, p_T) = S_L\, g_{1L}(x, p_T^2) - \frac{p_T \cdot S_T}{M}\, g_{1T}(x, p_T^2) \qquad (5.23)
$$

and so forth for the other functions. The first eight distributions of Eq. (5.22) are referred to as twist two, and the next 16 distributions are referred to as twist three, where we use the notion of "dynamical twist" as explained in Ref. [64]. The remaining eight functions of twist four have been omitted here and can be found in Ref. [58]. The 10 functions $f_{1T}^{\perp}$, $h_1^{\perp}$, e_L, e_T, $e_T^{\perp}$, f_T', $f_L^{\perp}$, $f_T^{\perp}$, $g^{\perp}$, h are T-odd [29, 58], i.e. they change sign under "naive time reversal", which is defined as usual time reversal, but without the interchange of initial and final states. The functions $g^{\perp}$ [24], $e_T^{\perp}$ and $f_T^{\perp}$ [58] exist because the direction n_- of the Wilson line. The nomenclature of the distribution functions follows closely that of Ref. [76], sometimes referred to as "Amsterdam notation."

We remark that a number of other notations exist for some of the distribution functions, see e.g. Refs. [25, 60, 81]. In particular, transverse-momentum-dependent functions at leading twist have been widely discussed by Anselmino et al. [9, 14, 15]. The connection between the notation in these papers and the one used here is discussed in App. C of Ref. [14].

We also list here the expressions for the traces of the correlator $\Phi(x, p_T)$ from Ref. [58]. With $\Phi^{[\Gamma]} = \frac{1}{2}\mathrm{Tr}[\Phi\,\Gamma]$ we have

$$
\Phi^{[\gamma^+]} = f_1 - \frac{\epsilon_T^{\rho\sigma} p_{T\rho} S_{T\sigma}}{M}\, f_{1T}^{\perp}\,, \qquad (5.24)
$$

$$
\Phi^{[\gamma^+\gamma_5]} = S_L\, g_{1L} - \frac{p_T \cdot S_T}{M}\, g_{1T}\,, \qquad (5.25)
$$

$$
\begin{aligned}
\Phi^{[\mathrm{i}\sigma^{\alpha+}\gamma_5]} &= S_T^{\alpha}\, h_1 + S_L\,\frac{p_T^{\alpha}}{M}\, h_{1L}^{\perp}\\
&\quad - \frac{p_T^{\alpha} p_T^{\rho} - \frac{1}{2}\, p_T^2\, g_T^{\alpha\rho}}{M^2}\, S_{T\rho}\, h_{1T}^{\perp} - \frac{\epsilon_T^{\alpha\rho} p_{T\rho}}{M}\, h_1^{\perp}\,, \qquad (5.26)
\end{aligned}
$$

$$
\Phi^{[1]} = \frac{M}{P^+}\Big[e - \frac{\epsilon_T^{\rho\sigma} p_{T\rho} S_{T\sigma}}{M}\, e_T^{\perp}\Big], \qquad (5.27)
$$

$$
\Phi^{[i\gamma_5]} = \frac{M}{P^+}\Big[S_L e_L - \frac{p_T \cdot S_T}{M}\, e_T\Big], \qquad (5.28)
$$

$$
\begin{aligned}
\Phi^{[\gamma^\alpha]} &= \frac{M}{P^+}\Big[- \epsilon_T^{\alpha\rho} S_{T\rho}\, f_T - S_L\,\frac{\epsilon_T^{\alpha\rho} p_{T\rho}}{M}\, f_L^{\perp}\\
&\quad - \frac{p_T^{\alpha} p_T^{\rho} - \frac{1}{2}\, p_T^2\, g_T^{\alpha\rho}}{M^2}\, \epsilon_{T\rho\sigma} S_T^{\sigma}\, f_T^{\perp} + \frac{p_T^{\alpha}}{M} f^{\perp}\Big], \qquad (5.29)
\end{aligned}
$$

$$\Phi^{[\gamma^\alpha\gamma_5]} = \frac{M}{P^+}\Big[S_T^{\,\alpha}\, g_T + S_L\, \frac{p_T^\alpha}{M}\, g_L^\perp - \frac{p_T^\alpha p_T^\rho - \frac{1}{2}\, p_T^2\, g_T^{\alpha\rho}}{M^2}\, S_{T\rho}\, g_T^\perp - \frac{\epsilon_T^{\alpha\rho}\, p_{T\rho}}{M}\, g^\perp\Big], \tag{5.30}$$

$$\Phi^{[i\sigma^{\alpha\beta}\gamma_5]} = \frac{M}{P^+}\Big[\frac{S_T^{\,\alpha} p_T^\beta - p_T^\alpha S_T^\beta}{M}\, h_T^\perp - \epsilon_T^{\alpha\beta}\, h\Big], \tag{5.31}$$

$$\Phi^{[i\sigma^{+-}\gamma_5]} = \frac{M}{P^+}\Big[S_L\, h_L - \frac{p_T\cdot S_T}{M}\, h_T\Big], \tag{5.32}$$

where α and β are restricted to be transverse indices. Here we made use of the combinations

$$f_T(x, p_T^2) = f_T'(x, p_T^2) - \frac{p_T^2}{2M^2}\, f_T^\perp(x, p_T^2), \tag{5.33}$$

$$g_T(x, p_T^2) = g_T'(x, p_T^2) - \frac{p_T^2}{2M^2}\, g_T^\perp(x, p_T^2), \tag{5.34}$$

$$h_1(x, p_T^2) = h_{1T}(x, p_T^2) - \frac{p_T^2}{2M^2}\, h_{1T}^\perp(x, p_T^2), \tag{5.35}$$

to separate off terms that vanish upon integration of the correlator over transverse momentum due to rotational symmetry. The conversion between the expressions with f_T and f_T' can be carried out using the identity

$$p_T^2\, \epsilon_T^{\alpha\rho} S_{T\rho} = p_T^\alpha\, \epsilon_T^{\rho\sigma} p_{T\rho} S_{T\sigma} + (p_T\cdot S_T)\, \epsilon_T^{\alpha\rho}\, p_{T\rho}\,, \tag{5.36}$$

which follows from the fact that there is no completely antisymmetric tensor of rank three in two dimensions.

Integrating the correlator over the transverse momentum p_T yields

$$\begin{aligned}\Phi(x) = \int \mathrm{d}^2\boldsymbol{p}_T\, \Phi(x, p_T) = {} & \frac{1}{2}\Big\{f_1\, \not{n}_+ + S_L g_1\, \gamma_5 \not{n}_+ + h_1\, \frac{[\not{S}_T, \not{n}_+]\gamma_5}{2}\Big\} \\ & + \frac{M}{2P^+}\Big\{e - \mathrm{i}\, S_L e_L \gamma_5 + f_T\, \epsilon_T^{\rho\sigma} S_{T\rho}\gamma_\sigma + g_T\, \gamma_5 \not{S}_T \\ & + S_L\, h_L\, \frac{[\not{n}_+, \not{n}_-]\gamma_5}{2} + \mathrm{i}\, h \frac{[\not{n}_+, \not{n}_-]}{2}\Big\},\end{aligned} \tag{5.37}$$

where the functions on the r.h.s. depend only on x and are given by

$$f_1(x) = \int \mathrm{d}^2\boldsymbol{p}_T\;\, f_1(x, p_T^2) \tag{5.38}$$

and so forth for the other functions. We have retained here one common exception of notation, namely

$$g_1(x) = \int \mathrm{d}^2\boldsymbol{p}_T\;\, g_{1L}(x, p_T^2). \tag{5.39}$$

Other notations are also in use for the leading-twist integrated functions, in particular $f_1^q = q$ (unpolarized distribution function), $g_1^q = \Delta q$ (helicity distribution function), $h_1^q = \delta q = \Delta_T q$ (transversity distribution function). The T-odd functions vanish due to time-reversal invariance [58]

$$\int \mathrm{d}^2\boldsymbol{p}_T\;\, f_T(x, p_T^2) = 0, \qquad \int \mathrm{d}^2\boldsymbol{p}_T\;\, e_L(x, p_T^2) = 0, \qquad \int \mathrm{d}^2\boldsymbol{p}_T\;\, h(x, p_T^2) = 0. \tag{5.40}$$

We have kept them in Eq. (5.37) so that one can readily obtain the analogous fragmentation correlator, where such functions do not necessarily vanish.

The fragmentation correlation function is defined as

$$\Delta_{ij}(z,k_T) = \frac{1}{2z}\sum_X \int \frac{d\xi^+\, d^2\boldsymbol{\xi}_T}{(2\pi)^3}\, e^{ik\cdot\xi}\, \langle 0|\,\mathcal{U}^{n_+}_{(+\infty,\xi)}\,\psi_i(\xi)|h,X\rangle\langle h,X|\bar{\psi}_j(0)\,\mathcal{U}^{n_+}_{(0,+\infty)}|0\rangle\Big|_{\xi^-=0}, \tag{5.41}$$

with $k^- = P_h^-/z$ and the Wilson lines

$$\mathcal{U}^{n_+}_{(+\infty,\xi)} \equiv \mathcal{U}^T(\infty_T,\boldsymbol{\xi}_T;+\infty^+)\,\mathcal{U}^{n_+}(+\infty^+,\xi^+;\boldsymbol{\xi}_T), \tag{5.42}$$

$$\mathcal{U}^{n_+}_{(0,+\infty)} \equiv \mathcal{U}^{n_+}(0^+,+\infty^+;\boldsymbol{0}_T)\,\mathcal{U}^T(\boldsymbol{0}_T,\infty_T;+\infty^+). \tag{5.43}$$

The notation $\mathcal{U}^{n_+}(a^+,b^+;\boldsymbol{c}_T)$ indicates a Wilson line running along the plus direction from $[0^-,a^+,\boldsymbol{c}_T]$ to $[0^-,b^+,\boldsymbol{c}_T]$, while $\mathcal{U}^T(\boldsymbol{a}_T,\boldsymbol{b}_T;c^+)$ indicates a gauge link running in the transverse direction from $[0^-,c^+,\boldsymbol{a}_T]$ to $[0^-,c^+,\boldsymbol{b}_T]$. The definition written above naturally applies for the correlation function appearing in e^+e^- annihilation. For semi-inclusive DIS it seems more natural to replace all occurrences of $+\infty^+$ in the gauge links by $-\infty^+$ [33]. However, in Ref. [48] it was shown that factorization can be derived in such a way that the fragmentation correlators in both semi-inclusive DIS and e^+e^- annihilation have gauge links pointing to $+\infty^+$.

The fragmentation correlation function (for a spinless or an unpolarized hadron) can be parameterized as

$$\begin{aligned}\Delta(z,k_T) = {}& \frac{1}{2}\left\{D_1\not{n}_- + iH_1^\perp\frac{[\not{k}_T,\not{n}_-]}{2M_h}\right\}\\ &+\frac{M_h}{2P_h^-}\left\{E + D^\perp\frac{\not{k}_T}{M_h} + iH\frac{[\not{n}_-,\not{n}_+]}{2} + G^\perp\gamma_5\frac{\epsilon_T^{\rho\sigma}\gamma_\rho k_{T\sigma}}{M_h}\right\},\end{aligned} \tag{5.44}$$

where the functions on the r.h.s. depend on z and k_T^2. To be complete, they should all carry also a flavor index, and the final hadron type should be specified. The correlation function Δ can be directly obtained from the correlation function Φ by changing[1]

$$n_+ \leftrightarrow n_-, \qquad \epsilon_T \to -\epsilon_T, \qquad P^+ \to P_h^-, \qquad M \to M_h, \qquad x \to 1/z, \tag{5.45}$$

and replacing the distribution functions with the corresponding fragmentation functions (f is replaced with D and all other letters are capitalized).

The correlator integrated over transverse momentum reads

$$\Delta(z) = z^2\int d^2\boldsymbol{k}_T\,\Delta(z,k_T) = D_1\frac{\not{n}_-}{2} + \frac{M_h}{2P_h^-}\left\{E + iH\frac{[\not{n}_-,\not{n}_+]}{2}\right\}, \tag{5.46}$$

where the functions on the r.h.s. are defined as

$$D_1(z) = z^2\int d^2\boldsymbol{k}_T\, D_1(z,k_T^2) \tag{5.47}$$

[1] The change of sign of the tensor ϵ_T is due to the exchange $n_+ \leftrightarrow n_-$ in its definition (10).

and so forth for the other functions. The prefactor z^2 appears because $D_1(z, k_T)$ is a probability density w.r.t. the transverse momentum $k'_T = -z k_T$ of the final-state hadron relative to the fragmenting quark. The fragmentation correlator for polarized spin-half hadrons is parameterized in analogy to Eqs. (5.22) and (5.37). As already remarked, in this case the functions D_T, E_L and H (the analogs of f_T, e_L and h) do not vanish because $|h, X\rangle$ is an interacting state that does not transform into itself under time-reversal. Of course, it should be taken as an outgoing state in the fragmentation correlator.

5.1.1 The quark-gluon-quark correlators

We now examine the quark-gluon-quark distribution correlation functions [33, 79]

$$\left(\Phi_D^\mu\right)_{ij}(x, p_T) = \int \frac{d\xi^- \, d^2\xi_T}{(2\pi)^3} \, e^{ip\cdot\xi} \langle P|\bar{\psi}_j(0)\, \mathcal{U}^{n_-}_{(0,+\infty)}\, \mathcal{U}^{n_-}_{(+\infty,\xi)}\, \mathrm{i}D^\mu(\xi)\, \psi_i(\xi)|P\rangle\Big|_{\xi^+=0}, \tag{5.48}$$

which contain the covariant derivative $\mathrm{i}D^\mu(\xi) = \mathrm{i}\partial^\mu + gA^\mu$. Using $\mathcal{U}^{n_-}_{(+\infty,\xi)}\, \mathrm{i}D^+(\xi)\, \psi(\xi) = \mathrm{i}\partial^+[\mathcal{U}^{n_-}_{(+\infty,\xi)}\, \psi(\xi)]$ we can write

$$\Phi_D^+(x, p_T) = x P^+ \Phi(x, p_T) \tag{5.49}$$

for the plus-component of the correlator. For the transverse components we define a further correlator [33]

$$\tilde{\Phi}_A^\alpha(x, p_T) = \Phi_D^\alpha(x, p_T) - p_T^\alpha\, \Phi(x, p_T), \tag{5.50}$$

which is manifestly gauge invariant. It reduces to a correlator defined as in Eq. (5.48) with the covariant derivative iD^μ replaced by gA_T^α if one has $\mathcal{U}^{n_-}_{(+\infty,\xi)} = 1$, which is the case in a light-cone gauge $A^+ = 0$ with suitable boundary conditions at light-cone infinity [26].

The quark-gluon-quark correlator can be related to the quark-quark correlator by means of the QCD equation of motion

$$[\mathrm{i}\not{D}(\xi) - m]\psi(\xi) = 0. \tag{5.51}$$

Let's work out an example here. If we multiply the above equation by γ^μ and use

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - \mathrm{i}\,\sigma^{\mu\nu}, \tag{5.52}$$

we obtain

$$(\mathrm{i}D^\mu + \mathrm{i}\sigma^{\nu\mu}\mathrm{i}D_\nu - m\gamma^\mu)\psi = 0, \tag{5.53}$$

In particular, if we choose $\mu = +$ and use the index α to denote only transverse components we have

$$\sigma^{\alpha+}\mathrm{i}D_\alpha\psi = \mathrm{i}\,(\mathrm{i}D^+ - \sigma^{+-}D_+ - m\gamma^+)\psi. \tag{5.54}$$

From the above relation, we can obtain (let's consider only the unpolarized case for the moment being)

$$\begin{aligned}
\mathrm{Tr}[\tilde{\Phi}_{A\alpha}\,\sigma^{\alpha+}] &= \mathrm{Tr}[\Phi_D^\alpha\,\sigma^{\alpha+}] - p_T^\alpha\,\mathrm{Tr}[\Phi\,\sigma^{\alpha+}] \\
&= \mathrm{i}\,\mathrm{Tr}[\Phi_D^+] - \mathrm{Tr}[\Phi_D^+\,\sigma^{+-}] - \mathrm{i}\,m\mathrm{Tr}[\Phi\,\gamma^+] - p_T^\alpha\,\mathrm{Tr}[\Phi\,\sigma^{\alpha+}] \\
&= \mathrm{i}\,xP^+\mathrm{Tr}[\Phi] - xP^+\mathrm{Tr}[\Phi\,\sigma^{+-}] - \mathrm{i}\,m\mathrm{Tr}[\Phi\,\gamma^+] - p_{T\alpha}\,\mathrm{Tr}[\Phi\,\sigma^{\alpha+}] \\
&= 2xM\,\mathrm{i}\,e + 2xMh - 2m\,\mathrm{i}\,f_1 - p_{T\alpha}\,\frac{2p_T^\alpha}{M}\,h_1^\perp
\end{aligned} \tag{5.55}$$

The important point is that the quark-gluon-quark correlator can be expressed in terms of the same distribution functions that we introduced in the quark-quark correlator (both at leading at subleading twist). No new functions are introduced. The more general case is treated below [Eq. (5.62) and ff.].

In general, the quark-gluon-quark correlation function can be decomposed as

$$\begin{aligned}
&\tilde{\Phi}_A^\alpha(x, p_T) = \\
&\frac{xM}{2}\Big\{\Big[(\tilde{f}^\perp - \mathrm{i}\tilde{g}^\perp)\frac{p_{T\rho}}{M} - (\tilde{f}'_T + \mathrm{i}\tilde{g}'_T)\,\epsilon_{T\rho\sigma}S_T^\sigma - (\tilde{f}_s^\perp + \mathrm{i}\,\tilde{g}_s^\perp)\frac{\epsilon_{T\rho\sigma}p_T^\sigma}{M}\Big](g_T^{\alpha\rho} - i\epsilon_T^{\alpha\rho}\gamma_5) \\
&\quad - (\tilde{h}_s + \mathrm{i}\,\tilde{e}_s)\gamma_T^\alpha\gamma_5 + \Big[(\tilde{h} + \mathrm{i}\,\tilde{e}) + (\tilde{h}_T^\perp - \mathrm{i}\,\tilde{e}_T^\perp)\frac{\epsilon_T^{\rho\sigma}p_{T\rho}S_{T\sigma}}{M}\Big]\mathrm{i}\gamma_T^\alpha + \ldots(g_T^{\alpha\rho} + \mathrm{i}\epsilon_T^{\alpha\rho}\gamma_5)\Big\}\frac{\not{n}_+}{2},
\end{aligned} \tag{5.56}$$

where the index α is restricted to be transverse here and in the following equations. The functions on the r.h.s. depend on x and p_T^2, except for the functions with subscript s, which are defined as in Eq. (5.23). The last term in the curly brackets is irrelevant for the construction of the hadronic tensor of semi-inclusive DIS and has not been parameterized explicitly. The only relevant traces of the quark-gluon-quark correlator are

$$\frac{1}{2Mx}\mathrm{Tr}[\tilde{\Phi}_{A\alpha}\,\sigma^{\alpha+}] = \tilde{h} + i\,\tilde{e} + \frac{\epsilon_T^{\rho\sigma}\,p_{T\rho}S_{T\sigma}}{M}\,(\tilde{h}_T^\perp - i\,\tilde{e}_T^\perp), \tag{5.57}$$

$$\frac{1}{2Mx}\mathrm{Tr}[\tilde{\Phi}_{A\alpha}\,\mathrm{i}\sigma^{\alpha+}\gamma_5] = S_L\,(\tilde{h}_L + i\,\tilde{e}_L) - \frac{p_T\cdot S_T}{M}\,(\tilde{h}_T + i\,\tilde{e}_T), \tag{5.58}$$

$$\begin{aligned}
\frac{1}{2Mx}\,\mathrm{Tr}[\tilde{\Phi}_{A\rho}(g_T^{\alpha\rho} + i\epsilon_T^{\alpha\rho}\gamma_5)\gamma^+] &= \frac{p_T^\alpha}{M}(\tilde{f}^\perp - i\tilde{g}^\perp) - \epsilon_T^{\alpha\rho}S_{T\rho}\,(\tilde{f}_T + i\tilde{g}_T) \\
&\quad - S_L\,\frac{\epsilon_T^{\alpha\rho}\,p_{T\rho}}{M}\,(\tilde{f}_L^\perp + i\,\tilde{g}_L^\perp) - \frac{p_T^\alpha\,p_T^\rho - \frac{1}{2}\,p_T^2\,g_T^{\alpha\rho}}{M^2}\,\epsilon_{T\rho\sigma}S_T^\sigma\,(\tilde{f}_T^\perp + i\tilde{g}_T^\perp),
\end{aligned} \tag{5.59}$$

where again we have used the combinations

$$\tilde{f}_T(x, p_T^2) = \tilde{f}'_T(x, p_T^2) - \frac{p_T^2}{2M^2}\,\tilde{f}_T^\perp(x, p_T^2), \tag{5.60}$$

$$\tilde{g}_T(x, p_T^2) = \tilde{g}'_T(x, p_T^2) - \frac{p_T^2}{2M^2}\,\tilde{g}_T^\perp(x, p_T^2). \tag{5.61}$$

The above traces have been given already in Ref. [24] for the terms without transverse polarization, whereas the terms with transverse polarization were partly discussed in Ref. [76] (the functions $e_T^\perp$ and $f_T^\perp$ introduced in Ref. [58] were missing).

Relations between correlation functions of different twist are provided by the equation of motion for the quark field

$$[\mathrm{i}\not{D}(\xi) - m]\psi(\xi) = [\gamma^+\mathrm{i}D^-(\xi) + \gamma^-\mathrm{i}D^+(\xi) + \gamma_T^\alpha\,\mathrm{i}D_\alpha(\xi) - m]\psi(\xi) = 0, \tag{5.62}$$

where m is the quark mass. To make their general structure transparent we decompose the correlators into terms of definite twist,

$$\Phi = \Phi_2 + \frac{M}{P^+}\,\Phi_3 + \left(\frac{M}{P^+}\right)^2\Phi_4, \qquad \frac{1}{M}\,\tilde{\Phi}_A^\alpha = \tilde{\Phi}_{A,3}^\alpha + \frac{M}{P^+}\,\tilde{\Phi}_{A,4}^\alpha + \left(\frac{M}{P^+}\right)^2\tilde{\Phi}_{A,5}^\alpha, \tag{5.63}$$

where the twist is indicated in the subscripts and the arguments (x, p_T) are suppressed for ease of writing. One has $\mathcal{P}_+\Phi_4 = \mathcal{P}_-\Phi_2 = 0$ and $\mathcal{P}_+\tilde{\Phi}^\alpha_{A,5} = \mathcal{P}_-\tilde{\Phi}^\alpha_{A,3} = 0$, where $\mathcal{P}_+ = \frac{1}{2}\gamma^-\gamma^+$ and $\mathcal{P}_- = \frac{1}{2}\gamma^+\gamma^-$ are the projectors on good and bad light-cone components, respectively [64]. Projecting Eq. (5.62) on its good components one obtains for the correlators

$$\mathcal{P}_+\Big[xM\,\gamma^-\Phi_3 + M\gamma_{T\rho}\tilde{\Phi}^\rho_{A,3} + \not{p}_T\,\Phi_2 - m\Phi_2\Big] \\ + \mathcal{P}_+\Big[xM\,\gamma^-\Phi_4 + M\gamma_{T\rho}\tilde{\Phi}^\rho_{A,4} + \not{p}_T\,\Phi_3 - m\Phi_3\Big]\frac{M}{P^+} = 0, \tag{5.64}$$

where the term with D^- has disappeared and the terms with D^+ and D^α have been replaced using Eqs. (5.49) and (5.50). Multiplying this relation with one of the matrices $\Gamma^+ = \{\gamma^+, \gamma^+\gamma_5, i\sigma^{\alpha+}\gamma_5\}$, which satisfy $\Gamma^+\mathcal{P}_+ = \Gamma^+(1-\mathcal{P}_-) = \Gamma^+$, and taking the trace gives

$$\mathrm{Tr}\Gamma^+\left[\gamma^- x\Phi_3 + \gamma_{T\rho}\tilde{\Phi}^\rho_{A,3} + \frac{\not{p}_T}{M}\Phi_2 - \frac{m}{M}\Phi_2\right] = 0, \tag{5.65}$$

where the terms multiplied by M/P^+ in Eq. (5.64) have disappeared because the trace of Dirac matrices cannot produce a term that transforms like P^+ under boosts in the light-cone direction. Inserting the parameterizations (5.22) and (5.56) into (5.65), one finds the following relations between T-even functions:

$$xe = x\tilde{e} + \frac{m}{M}\,f_1, \tag{5.66}$$

$$xf^\perp = x\tilde{f}^\perp + f_1, \tag{5.67}$$

$$xg'_T = x\tilde{g}'_T + \frac{m}{M}\,h_{1T}, \tag{5.68}$$

$$xg^\perp_T = x\tilde{g}^\perp_T + g_{1T} + \frac{m}{M}\,h^\perp_{1T}, \tag{5.69}$$

$$xg_T = x\tilde{g}_T - \frac{p_T^2}{2M^2}\,g_{1T} + \frac{m}{M}\,h_1, \tag{5.70}$$

$$xg^\perp_L = x\tilde{g}^\perp_L + g_{1L} + \frac{m}{M}\,h^\perp_{1L}, \tag{5.71}$$

$$xh_L = x\tilde{h}_L + \frac{p_T^2}{M^2}\,h^\perp_{1L} + \frac{m}{M}\,g_{1L}, \tag{5.72}$$

$$xh_T = x\tilde{h}_T - h_1 + \frac{p_T^2}{2M^2}\,h^\perp_{1T} + \frac{m}{M}\,g_{1T}, \tag{5.73}$$

$$xh^\perp_T = x\tilde{h}^\perp_T + h_1 + \frac{p_T^2}{2M^2}\,h^\perp_{1T}. \tag{5.74}$$

These relations can be found in Ref. [76], App. C. Neglecting quark-gluon-quark correlators (often referred to as the Wandzura-Wilczek approximation) is equivalent to setting all functions with a tilde to zero. For T-odd functions we have the following relations:

$$xe_L = x\tilde{e}_L, \tag{5.75}$$

$$xe_T = x\tilde{e}_T, \tag{5.76}$$

$$xe_T^\perp = x\tilde{e}_T^\perp + \frac{m}{M}\, f_{1T}^\perp, \tag{5.77}$$

$$xf_T' = x\tilde{f}_T' + \frac{p_T^2}{M^2}\, f_{1T}^\perp, \tag{5.78}$$

$$xf_T^\perp = x\tilde{f}_T^\perp + f_{1T}^\perp, \tag{5.79}$$

$$xf_T = x\tilde{f}_T + \frac{p_T^2}{2M^2}\, f_{1T}^\perp, \tag{5.80}$$

$$xf_L^\perp = x\tilde{f}_L^\perp, \tag{5.81}$$

$$xg^\perp = x\tilde{g}^\perp + \frac{m}{M}\, h_1^\perp, \tag{5.82}$$

$$xh = x\tilde{h} + \frac{p_T^2}{M^2}\, h_1^\perp. \tag{5.83}$$

Most of these relations can be found in Refs. [27, 30], and Eq. (5.82) can be inferred from Eq. (13) in Ref. [24]. Eqs. (5.77) and (5.79) have not been given before as they require the new functions introduced in Ref. [58]. We emphasize that the constraints due to the equations of motion remain valid in the presence of the appropriate Wilson lines in the correlation functions. All that is required for the gauge link $\mathcal{U}_{(0,\xi)}$ between the quark fields is the relation $\mathcal{U}_{(0,\xi)}\, \mathrm{i}D^+(\xi) = \mathrm{i}\partial^+ \mathcal{U}_{(0,\xi)}$ leading to Eq. (5.49). In contrast, the so-called Lorentz invariance relations used in earlier work are invalidated by the presence of the gauge links [57]. We remark that if quark-gluon-quark correlators are neglected, the time-reversal constraints (5.40) require that $\int \mathrm{d}^2p_T\, p_T^2\, f_{1T}^\perp(x, p_T^2) = 0$ and $\int \mathrm{d}^2p_T\, p_T^2\, h_1^\perp(x, p_T^2) = 0$.

Ex. 5.2

Check some of the relations above.

The quark-gluon-quark fragmentation correlator analogous to Φ_D is defined as

$$\left(\Delta_D^\mu\right)_{ij}(z, k_T) = \frac{1}{2z} \sum_X \int \frac{\mathrm{d}\xi^+\, \mathrm{d}^2\boldsymbol{\xi}_T}{(2\pi)^3}\, e^{\mathrm{i}k\cdot\xi}\, \langle 0|\, \mathcal{U}^{n_+}_{(+\infty,\xi)}\, iD^\mu(\xi)\, \psi_i(\xi)|h, X\rangle\langle h, X|\bar{\psi}_j(0)\, \mathcal{U}^{n_+}_{(0,+\infty)}|0\rangle\Big|_{\xi^-=0}. \tag{5.84}$$

The transverse correlator

$$\tilde{\Delta}_A^\alpha(z, k_T) = \Delta_D^\alpha(z, k_T) - k_T^\alpha\, \Delta(z, k_T) \tag{5.85}$$

can be decomposed as

$$\tilde{\Delta}_A^\alpha(z,k_T) = \frac{M_h}{2z}\Big\{(\tilde{D}^\perp - \mathrm{i}\,\tilde{G}^\perp)\,\frac{k_{T\rho}}{M_h}\,(g_T^{\alpha\rho} + i\epsilon_T^{\alpha\rho}\gamma_5) \\ + (\tilde{H} + \mathrm{i}\,\tilde{E})\,\mathrm{i}\gamma_T^\alpha + \ldots (g_T^{\alpha\rho} - i\epsilon_T^{\alpha\rho}\gamma_5)\Big\}\frac{\not{n}_-}{2}. \tag{5.86}$$

Using the equation of motion for the quark field, the following relations can be established between the functions appearing in the above correlator and the functions in the quark-quark correlator (5.44):

$$\frac{E}{z} = \frac{\tilde{E}}{z} + \frac{m}{M_h}\,D_1, \tag{5.87}$$

$$\frac{D^\perp}{z} = \frac{\tilde{D}^\perp}{z} + D_1, \tag{5.88}$$

$$\frac{G^\perp}{z} = \frac{\tilde{G}^\perp}{z} + \frac{m}{M_h}\,H_1^\perp, \tag{5.89}$$

$$\frac{H}{z} = \frac{\tilde{H}}{z} + \frac{k_T^2}{M_h^2}\,H_1^\perp. \tag{5.90}$$

5.1.2 Calculation of the hadronic tensor

We limit ourselves to the leading and first subleading term in the $1/Q$ expansion of the cross section and to graphs with the hard scattering at tree level. Loops can then only occur as shown in Fig. 5.1b, c, d, with gluons as external legs of the nonperturbative functions. The corresponding expression of the hadronic tensor is [33, 76]

$$2MW^{\mu\nu} = 2z\sum_a e_a^2 \int \mathrm{d}^2\boldsymbol{p}_T\;\mathrm{d}^2\boldsymbol{k}_T\;\delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T)\;\mathrm{Tr}\Big\{\Phi^a(x,p_T)\gamma^\mu\Delta^a(z,k_T)\gamma^\nu \\ - \frac{1}{Q\sqrt{2}}\Big[\gamma^\alpha\not{n}_+\gamma^\nu\tilde{\Phi}^a_{A\alpha}(x,p_T)\gamma^\mu\Delta^a(z,k_T) + \gamma^\alpha\not{n}_-\gamma^\mu\tilde{\Delta}^a_{A\alpha}(z,k_T)\gamma^\nu\Phi^a(x,p_T) + \text{h.c.}\Big]\Big\}, \tag{5.91}$$

with corrections of order $1/Q^2$, where the sum runs over the quark and antiquark flavors a, and e_a denotes the fractional charge of the struck quark or antiquark. The first, second and third term in Eq. (5.91) respectively correspond to the graphs in Fig. 5.1a, b and c, with gluons having transverse polarization. The analogs of Fig. 5.1b and c with the gluon on the other side of the final-state cut correspond to the “h.c.” terms in Eq. (5.91).

The Wilson lines needed in the color gauge invariant soft functions come from graphs with additional gluons exchanged between the hard scattering and either the distribution or the fragmentation function (as in Fig. 5.1b,c,d).

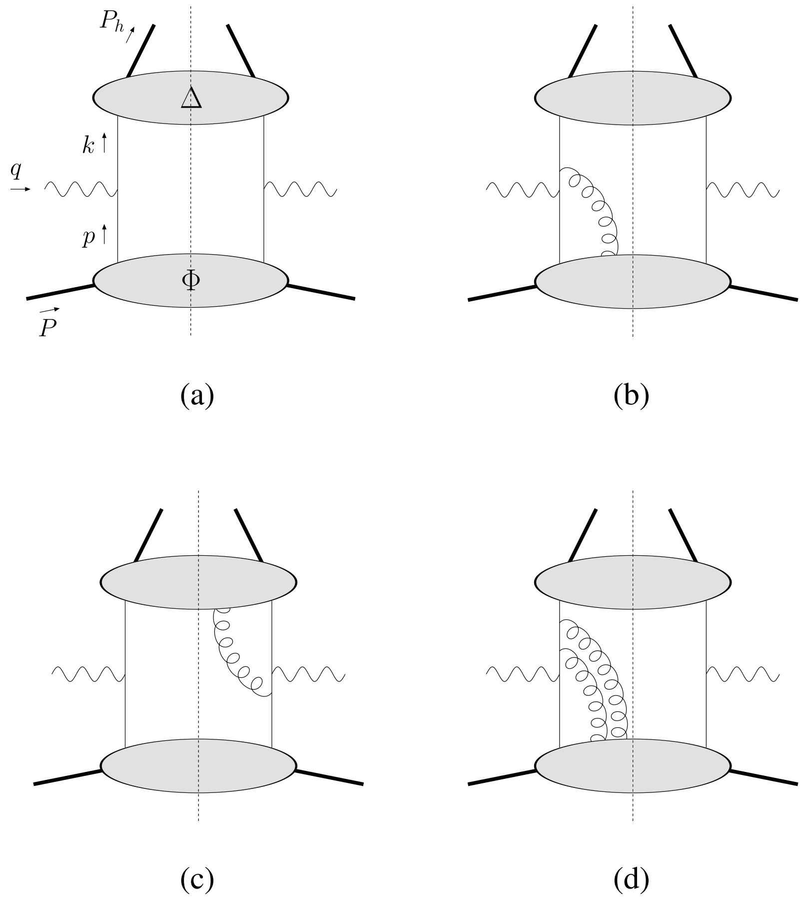


Figure 5.1. Examples of graphs contributing to semi-inclusive DIS at low transverse momentum of the produced hadron.

5.2 Frames

In the theoretical calculations we used a frame where P and P_h have no transverse components, and P points in the $+z$ direction. In reality, the choice of frame is implied by the fact that we assume $n_+ = [0, 1, \mathbf{0}_T]$ and $n_- = [1, 0, \mathbf{0}_T]$, and not by the above equations.

Experimentally, usually the frame of Fig. 1.1 is preferred, i.e. in general any frame where the proton and the photon have no transverse component, and the photon points in the $+z$ direction. The difference between the two frames is of order $1/Q$. This is why we did not worry too much about this issue in Ch. 1. Here however we have to worry about it. We'll follow the option of expressing all the vectors in the experimental frame.

In a frame where the proton and photon are collinear (but the proton still points in the $+z$ direction)

$$n_+ = [0,\ 1,\ \mathbf{0}_T], \tag{5.92}$$

$$n_- = \left[1,\ \frac{\boldsymbol{q}_T^2}{Q^2 - \boldsymbol{q}_T^2},\ -\frac{\boldsymbol{q}_T\sqrt{2}}{\sqrt{Q^2 - \boldsymbol{q}_T^2}}\right], \approx \left[1,\ 0,\ -\frac{\boldsymbol{q}_T\sqrt{2}}{Q}\right] \tag{5.93}$$

(the last approximation holds if $\boldsymbol{q}_T^2 \ll Q^2$).

The expression of the momenta in this frame are

$$P^\mu = \left[\frac{x_B M^2}{Q\sqrt{2}},\ \frac{Q}{x_B\sqrt{2}},\ \mathbf{0}\right] \tag{5.94a}$$

$$P_h^\mu = \left[\frac{z_h Q}{\sqrt{2}},\ \frac{M_h^2 + z_h^2 M_h^2}{z_h Q\sqrt{2}},\ -z_h \boldsymbol{q}_T\right] \tag{5.94b}$$

$$q^\mu = \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \tag{5.94c}$$

$$l^\mu = \left[\frac{Q}{y\sqrt{2}},\ \frac{(1-y)Q}{y\sqrt{2}},\ \frac{Q\sqrt{1-y}}{y},\ 0\right], \tag{5.94d}$$

$$l'^\mu = \left[\frac{(1-y)Q}{y\sqrt{2}},\ \frac{Q}{y\sqrt{2}},\ \frac{Q\sqrt{1-y}}{y},\ 0\right]. \tag{5.94e}$$

Usually, we identify

$$\boldsymbol{P}_{h\perp} = -z_h\, \boldsymbol{q}_T. \tag{5.95}$$

Note that in this change of frame transverse vectors also change, due to the change in the transverse projector

$$g_T^{\alpha\beta} = g^{\alpha\beta} - n_+^\alpha n_-^\beta - n_-^\alpha n_+^\beta \tag{5.96}$$

The rule is

$$a_T^\mu \approx \left[0,\ \frac{\boldsymbol{q}_T \cdot \boldsymbol{a}_T\sqrt{2}}{Q},\ \boldsymbol{a}_T\right] \tag{5.97}$$

There is a last step to do, namely inverting the direction of the z axis.

5.3 Results for structure functions

Inserting the parameterizations of the different correlators in the expression (5.91) of the hadronic tensor and using the equation-of-motion constraints just discussed, one can calculate the lepto-production cross section for semi-inclusive DIS and project out the different structure functions appearing in Eq. (1.32). To have a compact notation for the results, we introduce the unit vector $\hat{\boldsymbol{h}} = \boldsymbol{P}_{h\perp}/|\boldsymbol{P}_{h\perp}|$ and the notation

$$C[w\, f\, D] = x_B \sum_a e_a^2 \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^{(2)}(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z)\, w(\boldsymbol{p}_T, \boldsymbol{k}_T)\, f^a(x_B, p_T^2)\, D^a(z, k_T^2), \tag{5.98}$$

where $w(\boldsymbol{p}_T, \boldsymbol{k}_T)$ is an arbitrary function and the summation runs over quarks and antiquarks. The expressions for the structure functions appearing in Eq. (1.32) are

$$F_{UU,T} = \mathcal{C}\big[f_1 D_1\big], \tag{5.99}$$

$$F_{UU,L} = 0, \tag{5.100}$$

$$F_{UU}^{\cos\phi_h} = \frac{2M}{Q}\,\mathcal{C}\Big[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T}{M_h}\Big(x_B h\, H_1^{\perp} + \frac{M_h}{M}\, f_1 \frac{\tilde{D}^{\perp}}{z}\Big) - \frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M}\Big(x_B f^{\perp} D_1 + \frac{M_h}{M}\, h_1^{\perp}\frac{\tilde{H}}{z}\Big)\Big], \tag{5.101}$$

$$F_{UU}^{\cos 2\phi_h} = \mathcal{C}\Big[-\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T) - \boldsymbol{k}_T\cdot\boldsymbol{p}_T}{M M_h} h_1^{\perp} H_1^{\perp}\Big], \tag{5.102}$$

$$F_{LU}^{\sin\phi_h} = \frac{2M}{Q}\,\mathcal{C}\Big[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T}{M_h}\Big(x_B e\, H_1^{\perp} + \frac{M_h}{M}\, f_1 \frac{\tilde{G}^{\perp}}{z}\Big) + \frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M}\Big(x_B g^{\perp} D_1 + \frac{M_h}{M}\, h_1^{\perp}\frac{\tilde{E}}{z}\Big)\Big], \tag{5.103}$$

$$F_{UL}^{\sin\phi_h} = \frac{2M}{Q}\,\mathcal{C}\Big[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T}{M_h}\Big(x_B h_L H_1^{\perp} + \frac{M_h}{M}\, g_{1L} \frac{\tilde{G}^{\perp}}{z}\Big) + \frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M}\Big(x_B f_L^{\perp} D_1 - \frac{M_h}{M}\, h_{1L}^{\perp}\frac{\tilde{H}}{z}\Big)\Big], \tag{5.104}$$

$$F_{UL}^{\sin 2\phi_h} = \mathcal{C}\Big[-\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T) - \boldsymbol{k}_T\cdot\boldsymbol{p}_T}{M M_h} h_{1L}^{\perp} H_1^{\perp}\Big], \tag{5.105}$$

$$F_{LL} = \mathcal{C}\big[g_{1L} D_1\big], \tag{5.106}$$

$$F_{LL}^{\cos\phi_h} = \frac{2M}{Q}\,\mathcal{C}\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T}{M_h}\Big(x_B e_L H_1^{\perp} - \frac{M_h}{M}\, g_{1L} \frac{\tilde{D}^{\perp}}{z}\Big) - \frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M}\Big(x_B g_L^{\perp} D_1 + \frac{M_h}{M}\, h_{1L}^{\perp}\frac{\tilde{E}}{z}\Big)\Big], \tag{5.107}$$

$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\Big[-\frac{\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\Big], \tag{5.108}$$

$$F_{UT,L}^{\sin(\phi_h-\phi_S)} = 0, \tag{5.109}$$

$$F_{UT}^{\sin(\phi_h+\phi_S)} = \mathcal{C}\Big[-\frac{\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{k}_T}{M_h} h_1 H_1^{\perp}\Big], \tag{5.110}$$

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = \mathcal{C}\Big[\frac{2\,(\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{p}_T)\,(\boldsymbol{p}_T\cdot\boldsymbol{k}_T) + \boldsymbol{p}_T^2\,(\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{k}_T) - 4\,(\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{p}_T)^2\,(\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{k}_T)}{2M^2 M_h}\, h_{1T}^{\perp} H_1^{\perp}\Big], \tag{5.111}$$

$$\begin{aligned} F_{UT}^{\sin\phi_S} = {}& \frac{2M}{Q}\,\mathcal{C}\Big\{\Big(x_B f_T D_1 - \frac{M_h}{M}\, h_1 \frac{\tilde{H}}{z}\Big) \\ & - \frac{\boldsymbol{k}_T\cdot\boldsymbol{p}_T}{2M M_h}\Big[\Big(x_B h_T H_1^{\perp} + \frac{M_h}{M} g_{1T}\,\frac{\tilde{G}^{\perp}}{z}\Big) - \Big(x_B h_T^{\perp} H_1^{\perp} - \frac{M_h}{M}\, f_{1T}^{\perp}\,\frac{\tilde{D}^{\perp}}{z}\Big)\Big]\Big\}, \end{aligned} \tag{5.112}$$

$$\begin{aligned} F_{UT}^{\sin(2\phi_h-\phi_S)} = {}& \frac{2M}{Q}\,\mathcal{C}\Big\{\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)^2 - \boldsymbol{p}_T^2}{2M^2}\Big(x_B f_T^{\perp} D_1 - \frac{M_h}{M}\, h_{1T}^{\perp}\frac{\tilde{H}}{z}\Big) \\ & - \frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T) - \boldsymbol{k}_T\cdot\boldsymbol{p}_T}{2M M_h}\Big[\Big(x_B h_T H_1^{\perp} + \frac{M_h}{M} g_{1T}\,\frac{\tilde{G}^{\perp}}{z}\Big) \\ & + \Big(x_B h_T^{\perp} H_1^{\perp} - \frac{M_h}{M}\, f_{1T}^{\perp}\,\frac{\tilde{D}^{\perp}}{z}\Big)\Big]\Big\}, \end{aligned} \tag{5.113}$$

$$F_{LT}^{\cos(\phi_h-\phi_S)} = \mathcal{C}\Big[\frac{\hat{\boldsymbol{P}}_{h\perp}\cdot\boldsymbol{p}_T}{M} g_{1T} D_1\Big], \tag{5.114}$$

$$\begin{aligned} F_{LT}^{\cos\phi_S} &= \frac{2M}{Q}\,\mathcal{C}\Big\{-\Big(x_B g_T D_1 + \frac{M_h}{M}\,h_1\,\frac{\tilde{E}}{z}\Big) \\ &\quad + \frac{\boldsymbol{k}_T\cdot\boldsymbol{p}_T}{2MM_h}\Big[\Big(x_B e_T H_1^{\perp} - \frac{M_h}{M} g_{1T}\,\frac{\tilde{D}^{\perp}}{z}\Big) + \Big(x_B e_T^{\perp} H_1^{\perp} + \frac{M_h}{M}\,f_{1T}^{\perp}\,\frac{\tilde{G}^{\perp}}{z}\Big)\Big]\Big\}, \end{aligned} \tag{5.115}$$

$$\begin{aligned} F_{LT}^{\cos(2\phi_h-\phi_S)} &= \frac{2M}{Q}\,\mathcal{C}\Big\{-\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)^2 - \boldsymbol{p}_T^2}{2M^2}\Big(x_B g_T^{\perp} D_1 + \frac{M_h}{M}\,h_{1T}^{\perp}\,\frac{\tilde{E}}{z}\Big) \\ &\quad + \frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T) - \boldsymbol{k}_T\cdot\boldsymbol{p}_T}{2MM_h}\Big[\Big(x_B e_T H_1^{\perp} - \frac{M_h}{M} g_{1T}\,\frac{\tilde{D}^{\perp}}{z}\Big) \\ &\quad - \Big(x_B e_T^{\perp} H_1^{\perp} + \frac{M_h}{M}\,f_{1T}^{\perp}\,\frac{\tilde{G}^{\perp}}{z}\Big)\Big]\Big\}. \end{aligned} \tag{5.116}$$

Notice that distribution and fragmentation functions do not appear in a symmetric fashion in these expressions: there are only twist-three fragmentation functions with a tilde and only twist-three distribution functions without tilde. This asymmetry is not surprising because in Eq. (1.32) the structure functions themselves are introduced in an asymmetric way, with azimuthal angles referring to the axis given by the four-momenta of the target nucleon and the photon, rather than of the target nucleon and the detected hadron.

A few comments concerning the comparison with the existing literature are in order here. First of all, it has to be stressed that in much of the past literature a different definition of the azimuthal angles has been used, whereas in the present work we adhere to the Trento conventions [22]. To compare with those papers, the signs of ϕ_h and of ϕ_S have to be reversed. The terms with the distribution functions $f_T^{\perp}$ and $e_T^{\perp}$ have not been given before. All leading-twist structure functions here are consistent with those given in Eqs. (36) and (37) of Ref. [32] when only photon exchange is taken into consideration. The structure functions $F_{LU}^{\sin\phi_h}$ and $F_{UL}^{\sin\phi_h}$ in our Eqs. (5.103) and (5.104) correspond to Eqs. (16) and (25) in Ref. [24]. The other six twist-three structure functions were partially given in the original work of Mulders and Tangerman [76], but excluding T-odd distribution functions, assuming Gaussian transverse-momentum distributions, and without the contributions from the fragmentation function $G^{\perp}$.

The structure function $F_{UU}^{\cos\phi_h}$ is associated with the so-called Cahn effect [42, 43]. If one neglects the quark-gluon-quark functions $\tilde{D}^{\perp}$ in Eq. (5.101) and $\tilde{f}^{\perp}$ in Eq. (5.67) as well as the T-odd distribution functions h and $h_1^{\perp}$, our result becomes

$$F_{UU}^{\cos\phi_h} \approx \frac{2M}{Q}\,\mathcal{C}\Big[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M}\,f_1 D_1\Big]. \tag{5.117}$$

This coincides with the $\cos\phi_h$ term calculated to order $1/Q$ in the parton model with intrinsic transverse momentum included in distribution and fragmentation functions, see e.g. Eqs. (32) and (33) in Ref. [13].

Let us briefly mention some experimental results and phenomenological analyses for the structure functions given above. For simplicity we do not distinguish between measurements of the structure functions and of the associated spin or angular asymmetries, which correspond to the ratio of the appropriate structure functions and $F_{UU,T} + \epsilon F_{UU,L}$.

1. Measurements of the cross-section components containing the structure function $F_{UU}^{\cos\phi_h}$ have been reported in Refs. [2, 16, 37, 44]. A description of the $\cos\phi_h$ modulation by the Cahn effect alone has been given in Ref. [13]. The same analysis can be applied to the structure function $F_{LL}^{\cos\phi_h}$, leading to the results of Ref. [10].

2. $F_{UU}^{\cos 2\phi_h}$ contains the functions $h_1^\perp$ (Boer-Mulders function [29]) and $H_1^\perp$ (Collins function [46]). It has been measured in Refs. [37, 44].

3. The structure function $F_{LU}^{\sin\phi_h}$ has been recently measured by the CLAS collaboration [17].

4. The structure function $F_{UL}^{\sin\phi_h}$ has been measured by HERMES [7]. The precise extraction of this observable requires care because in experiments the target is polarized along the direction of the lepton beam and not of the virtual photon [24, 50, 69, 77]. This implies that the longitudinal target-spin asymmetries measured in Refs. [3–5] receive contributions not only from $F_{UL}^{\sin\phi_h}$, but at the same order in $1/Q$ also from $F_{UT,T}^{\sin(\phi_h-\phi_S)}$ and $F_{UT}^{\sin(\phi_h+\phi_S)}$ (see also the phenomenological studies of Refs. [51–55, 74, 82]). In Ref. [7] the HERMES collaboration has separated the different contributions to the experimental $\sin\phi_h$ asymmetry with longitudinal target polarization and shown that $F_{UL}^{\sin\phi_h}$ is dominant in the kinematics of the measurement.

5. $F_{UT,T}^{\sin(\phi_h-\phi_S)}$ contains the Sivers function [83] and has been recently measured for a proton target at HERMES [6] and for a deuteron target at COMPASS [8]. Extractions of the Sivers function from the experimental data were performed in Refs. [12, 45, 86] (see Ref. [11] for a comparison of the various extractions).

6. The structure function $F_{UT}^{\sin(\phi_h+\phi_S)}$ contains the transversity distribution function [25, 81] and the Collins function. As the previous structure function, it has been measured by HERMES [6] on the proton and by COMPASS [8] on the deuteron. Phenomenological studies have been presented in Ref. [86], where information about the Collins function was extracted, and in Ref. [56], where constraints on the transversity distribution function were obtained by using additional information from a Collins asymmetry measured in e^+e^- annihilation [1].

Integration of Eqs. (5.99) to (5.116) over the transverse momentum $\boldsymbol{P}_{h\perp}$ of the outgoing hadron leads to the following expressions for the integrated structure functions in Eq. (1.38):

$$F_{UU,T} = x_B \sum_a e_a^2\, f_1^a(x_B)\, D_1^a(z), \tag{5.118}$$

$$F_{UU,L} = 0, \tag{5.119}$$

$$F_{LL} = x_B \sum_a e_a^2\, g_1^a(x_B)\, D_1^a(z), \tag{5.120}$$

$$F_{UT}^{\sin\phi_S} = -x_B \sum_a e_a^2\, \frac{2M_h}{Q}\, h_1^a(x_B)\, \frac{\tilde{H}^a(z)}{z}, \tag{5.121}$$

$$F_{LT}^{\cos\phi_S} = -x_B \sum_a e_a^2 \frac{2M}{Q} \Big(x_B g_T^a(x_B) D_1^a(z) + \frac{M_h}{M}\, h_1^a(x_B)\, \frac{\tilde{E}^a(z)}{z} \Big). \tag{5.122}$$

Finally, the structure functions for totally inclusive DIS can be obtained from Eqs. (1.40) and (1.41) to (1.44). This gives the standard results [76]

$$F_1 = \frac{1}{2} \sum_a e_a^2\ f_1^a(x_B), \tag{5.123}$$

$$F_L = 0, \tag{5.124}$$

$$g_1 = \frac{1}{2} \sum_a e_a^2\ g_1^a(x_B), \tag{5.125}$$

$$g_1 + g_2 = \frac{1}{2} \sum_a e_a^2\ g_T^a(x_B), \tag{5.126}$$

where given the accuracy of our calculation we have replaced $g_1 - \gamma^2 g_2$ by g_1 in (5.125), and where we have used

$$\sum_h \int \mathrm{d}z\, z\, D_1^a(z) = 1, \qquad \sum_h \int \mathrm{d}z\, \tilde{E}^a(z) = 0, \qquad \sum_h \int \mathrm{d}z\, \tilde{H}^a(z) = 0. \tag{5.127}$$

The first relation is the well-known momentum sum rule for fragmentation functions. The second relation was already pointed out in Ref. [63]. The sum rule for $\tilde{H}$ follows from Eq. (5.121) and the time-reversal constraint (1.45).

5.4 Semi-inclusive jet production

In this appendix, we take into consideration the process

$$\ell(l) + N(P) \to \ell(l') + \mathrm{jet}(P_j) + X \tag{5.128}$$

in the kinematical limit of large Q^2 at fixed x and $P_{h\perp}^2$. In the context of our tree-level calculation, we identify the jet with the quark scattered from the virtual photon. We then have $z = 1$ and the cross section formula is identical to Eq. (1.32), except that it is not differential in z. Correspondingly, the structure functions do not depend on this variable. The structure functions for the process (5.128) can be obtained from those of one-particle inclusive DIS in Eqs. (5.99) to (5.116) by replacing $D_1(z, k_T^2)$ with $\delta(1-z)\delta^{(2)}(\boldsymbol{k}_T)$, setting all other fragmentation functions to zero and integrating over z. This gives

$$F_{UU,T} = x_B \sum_a e_a^2\, f_1^a(x, P_{j\perp}^2), \tag{5.129}$$

$$F_{UU}^{\cos\phi_h} = -x_B \sum_a e_a^2\, \frac{2|\boldsymbol{P}_{j\perp}|}{Q}\, x_B f^{\perp a}(x, P_{j\perp}^2), \tag{5.130}$$

$$F_{LU}^{\sin\phi_h} = x_B \sum_a e_a^2\, \frac{2|\boldsymbol{P}_{j\perp}|}{Q}\, x_B g^{\perp a}(x, P_{j\perp}^2), \tag{5.131}$$

$$F_{UL}^{\sin\phi_h} = x_B \sum_a e_a^2\, \frac{2|\boldsymbol{P}_{j\perp}|}{Q}\, x_B f_L^{\perp a}(x, P_{j\perp}^2), \tag{5.132}$$

$$F_{LL} = x_B \sum_a e_a^2\, g_{1L}^a(x, P_{j\perp}^2), \tag{5.133}$$

$$F_{LL}^{\cos\phi_h} = -x_B \sum_a e_a^2\, \frac{2|\boldsymbol{P}_{j\perp}|}{Q}\, x_B g_L^{\perp a}(x, P_{j\perp}^2), \tag{5.134}$$

$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = -x_B \sum_a e_a^2\, \frac{|\boldsymbol{P}_{j\perp}|}{M}\, f_{1T}^{\perp a}(x, P_{j\perp}^2), \tag{5.135}$$

$$F_{UT}^{\sin\phi_S} = x_B \sum_a e_a^2\, \frac{2M}{Q}\, x_B f_T^a(x, P_{j\perp}^2), \tag{5.136}$$

$$F_{UT}^{\sin(2\phi_h-\phi_S)} = x_B \sum_a e_a^2\, \frac{|\boldsymbol{P}_{j\perp}|^2}{MQ}\, x_B f_T^{\perp a}(x, P_{j\perp}^2), \tag{5.137}$$

$$F_{LT}^{\cos(\phi_h-\phi_S)} = x_B \sum_a e_a^2\, \frac{|\boldsymbol{P}_{j\perp}|}{M}\, g_{1T}^a(x, P_{j\perp}^2), \tag{5.138}$$

$$F_{LT}^{\cos\phi_S} = -x_B \sum_a e_a^2\, \frac{2M}{Q}\, x_B g_T^a(x, P_{j\perp}^2), \tag{5.139}$$

$$F_{LT}^{\cos(2\phi_h-\phi_S)} = -x_B \sum_a e_a^2\, \frac{|\boldsymbol{P}_{j\perp}|^2}{MQ}\, x_B g_T^{\perp a}(x, P_{j\perp}^2), \tag{5.140}$$

whereas the remaining 6 structure functions are zero. The results for the terms with indices UL and LU correspond to those in Ref. [24]. Integration of the cross section over $\boldsymbol{P}_{h\perp}$ leads to the results for inclusive DIS in Eqs. (5.123) to (5.126). Most terms vanish due to the angular integration, and $F_{UT}^{\sin\phi_S}$ in Eq. (5.136) vanishes due to the time-reversal condition (5.40).

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