The compression function for the fluorescence detector of AUGER

Paolo Walter Cattaneo

INFN Sezione di Pavia and Dipartimento di Fisica Nucleare e Teorica, Via Bassi 6, Pavia, I-27100 Italy
Email: Paolo.Cattaneo@pv.infn.it

Abstract We study the two linear branch compression function matching the requirements of the fluorescence detector of AUGER. The parameters of the compression curve, the slope ratio and the break point are calculated to guarantee a minimal additional error due to quantization.

1 Detector resolution and dynamic range

The approach used in the following has been firstly developed in [2]-[3]-[4]. We assume that the resolution of the detector is of the same form of the sampling calorimeters with the number of photoelectrons n replacing the energy

$$\sigma(n) = \sqrt{b^2 + a^2 n + c^2 n^2}, \tag{1}$$

where a the constant governing the fluctuation of photoelectrons, b the noise and c the constant term.
For the AUGER fluorescence detector, reasonable values may be $a = 1.$ (pure Poisson fluctuation on the number of photoelectrons), $b = 2.7$ is the sky noise estimation and $c = 0.05$ (5%) is an estimation (probably optimistic) of the intrinsic capability of the system in measuring energy.
The problem in having a linear ADC is that the quantization error is constant and for a growing absolute error is too good at large ADC counts. Said otherwise the LSB is not matched to the detector resolution.
We can, on the opposite, require that the LSB be exactly matched with the detector resolution, that is requiring

$$LSB(n) = \frac{\sigma(n)}{k}, \tag{2}$$

where k measures how much the original resolution can be accepted to be decreased because of the quantization error. We can in fact write the quantization error $\sigma_q(n) = \frac{LSB(n)}{\sqrt{12}}$ and the total error as $\sigma_t(n) = \sigma_q(n) \oplus \sigma(n)$ and define

$$R = \frac{\sigma_t(n) - \sigma(n)}{\sigma(n)} \Longrightarrow R = \sqrt{1 + \frac{1}{12k^2}} - 1. \tag{3}$$

Defining $N(n)$ as the number of ADC counts corresponding to n photoelectrons, we have by definition

$$\begin{aligned} \frac{dN(n)}{dn} &= \frac{1}{LSB(n)} = k\frac{1}{\sigma(n)} \\ N(n) &= \int dn \frac{k}{\sigma(n)} = \frac{k}{c}\ln(a^2 + 2c^2 n + 2c\sigma(n)) + A, \end{aligned} \tag{4}$$

where A is a constant fixed by the requirements on the dynamic range.
Eq.4 gives the compression function required to obtain a fixed quantization error contribution to the resolution. This function is difficult to implement circuitaly and will not be pursued any further.
We will consider more closely the possibility of implementing the compression function through a two branch linear transfer function.
We make the following definitions: n_i is the input range expressed in bits, n_o is the number of bits of the ADC chip, n^{lsb} is the number of photoelectrons corresponding to an LSB in the first branch, n^{max} the photoelectron number corresponding to the ADC saturation, α the ratio between the slope of the first and the second branches and n^b the branch point where the two slopes change.
The quantization error is $\sigma_q(n) = \frac{n^{lsb}}{\sqrt{12}}$ for $n < n^b$ and $\sigma_q(n) = \alpha\frac{n^{lsb}}{\sqrt{12}}$ for $n \geq n^b$.
The maximum quantization errors are always at the beginning of each branch and the output range is saturated by $n^{max} = 2^{n_i} n^{lsb}$ photoelectrons

$$\begin{aligned} n^{lsb} &\leq \frac{1}{k} b \\ n^{lsb}\alpha &\leq \frac{1}{k}\sqrt{b^2 + a^2 n^b + c^2 n^{b2}} \\ 2^{n_o} &= \frac{n^b}{n^{lsb}} + \frac{n^{max} - n^b}{n^{lsb}\alpha}. \end{aligned} \tag{5}$$

1.1 Numerical solution

The system in Eq.5 is complicated to solve analytically so that an effective solutions can be found numerically. We can fix values for $\frac{1}{k}$, or equivalently R, n_i and n_o and search solutions for n^{lsb}, α, n^b.
For convenience we report the result in terms of $y = n^b/n^{lsb}$, that is the number of ADC counts corresponding to the break point and the maximum value of n^{lsb} satisfying the first equation of Eq.5 has been chosen.
It is easy to demonstrate that the solutions α, n^b satisfying the unequality of Eq.5, if they exist, are intervals delimited by minimal and maximal values for α and n^b. Solving for α, the last two members of the unequality Eq.5 can be casted as

$$\frac{1}{k}\sqrt{\frac{b^2}{n^{lsb2}} + \frac{a^2}{n^{lsb}}y + c^2y^2} \geq \frac{2^{n_i} - y}{2^{n_o} - y}. \tag{6}$$

In Fig.1 the two sides of Eq.6, representing α, are plotted for $R = 0.1$, $n_o = 12$ and $n_i \in (15, 20)$ are plotted. The unequality has solutions for $n_i \leq 18$. Requiring $R = 0.2$ just marginally brings in the $n_i = 19$ case.
No solution except, marginally, for $n_i = 15$ is available for $n_o = 10$.

1.2 Exact analytical solution

Eq.6 can be solved analytically squaring both sides and bringing the resulting fourth order equation to the normal form $y^4 + a_1y^3 + a_2y^2 + a_3y + a_4$

$$y^4 + \left(\frac{a^2}{c^2n^{lsb}}\right)y^3 + \left(\frac{b^2}{c^2n^{lsb2}} - \frac{2^{n_o+1}}{n^{lsb}}\frac{a^2}{c^2} + 2^{n_o} - \frac{k^2}{c^2}\right)y^2 + \tag{7}$$

$$\left(\frac{2^{2n_o}}{n^{lsb}}\frac{a^2}{c^2} - \frac{2^{n_o+1}}{n^{lsb2}}\frac{b^2}{c^2} + 2^{n_i+1}\frac{k^2}{c^2}\right)y + \left(\frac{2^{n_o}}{n^{lsb2}}\frac{b^2}{c^2} - 2^{2n_i}\frac{k^2}{c^2}\right) = 0. \tag{8}$$

This equation can be solved using the classical approach through its cubic resolvent. Defining $p = a_2 - \frac{3}{8}a_1^2$, $q = \frac{1}{8}a_1^3 - \frac{1}{2}a_1a_2 + a_3$ and $r = -\frac{3}{256}a_1^4 + \frac{1}{16}a_1^2a_2 - \frac{1}{4}a_1a_3 + a_4$, its resolvent is

$$\Delta = 16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r + 256r^3 - 27q^4. \tag{9}$$

Because a graphical study of the equation in the form Eq.6 shows that there are always two real solution with unphysical values ($y < 0$ or $y > 2^{n_o}$), and

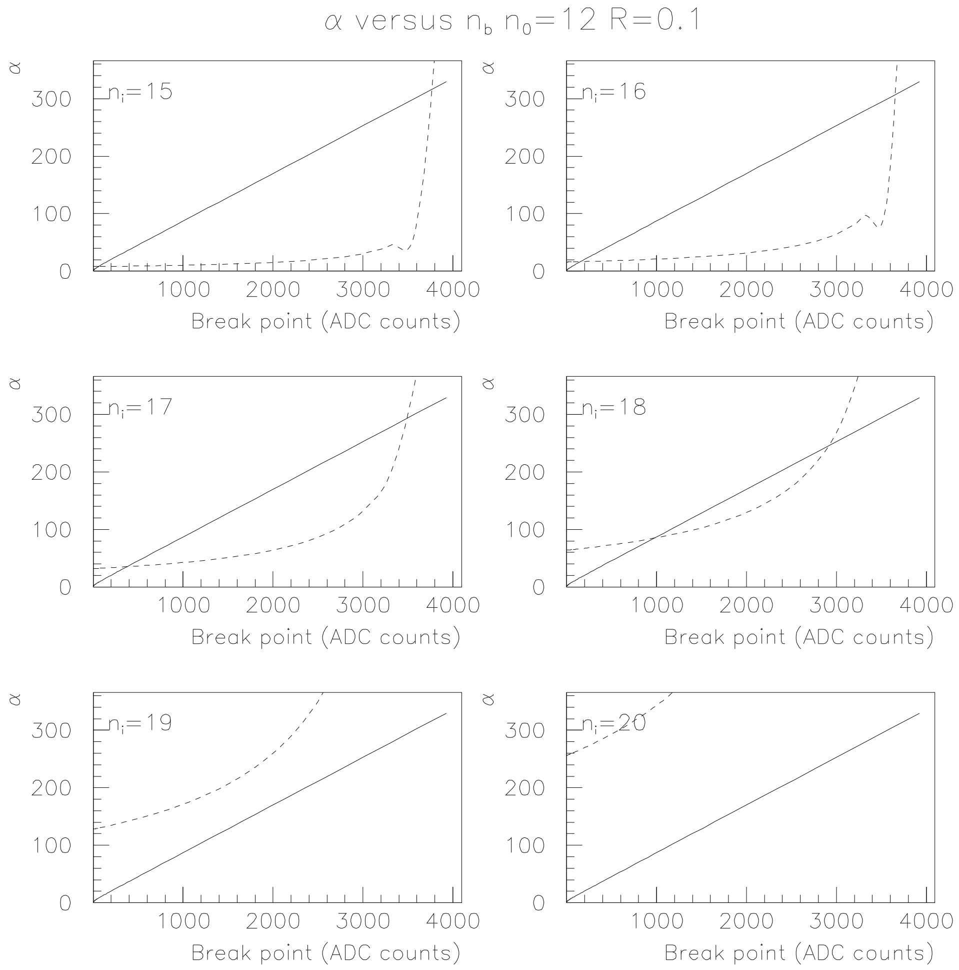

Fig. 1. Numerical determination of α and $\frac{n^b}{n^{lsb}}$ for R=0.1

two others are possible in the region $0 < y < 2^{n_o}$, if $\Delta < 0$, there are no physical solutions, if $\Delta \geq 0$ there are two real solutions $0 < y_{min} \leq y_{max} < 2^{n_o}$. The values $y_{min} \leq y \leq y_{max}$ satisfies the unequality of Eq.6.

1.3 Approximate analytical solution

It is also possible to simplify the analytical solution of the system of Eq.5 applying some approximation to Eq.6.
We assume certain relations on the values of the resolution parameters and of $\frac{1}{k}$, namely: the contribution $\frac{b}{n^{lsb}}$ be negligible compared to the other terms for $y = \frac{n_b}{n^{lsb}}$. The terms under the radix are grouped so to write it as $\sqrt{1+x} \approx 1 + x/2$, the approximation is valid for $(\frac{a}{c})^2 \frac{1}{n^{lsb}} \leq \frac{n_b}{n^{lsb}}$. The validity of the

approximation can be checked a posteriori.
After some calculations, within the prescribed approximations, the relevant equation is

$$y^2 + y\left(\frac{a^2}{2c^2 n^{lsb}} - 2^{n_o} - \frac{k}{c}\right) + \left(2^{n_i}\frac{k}{c} - \frac{a^2 2^{n_o}}{2n^{lsb}c^2}\right) \leq 0. \tag{10}$$

From this equation we require the determinant to be positive to establish the bound on n_i

$$\left(\frac{a^2 k}{c^2 n^{lsb}} - 2^{n_o} - \frac{k}{c}\right)^2 - \left(\frac{4k}{c}2^{n_i} - 2\frac{a^2 2^{n_o}}{n^{lsb}}\right) > 0$$

$$log_2\left[\frac{c}{4k}\left(\left(\frac{a^2}{2c^2 n^{lsb}} - 2^{n_o} - \frac{k}{c}\right)^2 + 2\frac{a^2 2^{n_o}}{c^2 n^{lsb}}\right)\right] > n_i \tag{11}$$

Inserting the values for the parameters used for Fig.1, the result is $n_i > 18.2$, that confirm the numerical calculation.
If Eq.10 has solutions, they are

$$y_{2,1} = \frac{1}{2}\left[2^{n_o} + \frac{k}{c} - \frac{a^2}{2c^2 n^{lsb}} \pm \sqrt{\left(\frac{a^2}{2c^2 n^{lsb}} - 2^{n_o} - \frac{k}{c}\right)^2 - \left(\frac{4k}{c}2^{n_i} - \frac{4a^2 2^{n_o}}{n^{lsb}c^2}\right)}\right]. \tag{12}$$

The condition of Eq.10 is satisfied for $y_1 \leq y \leq y_2$.

1.4 Error minimization

In this interval, we can calculate the value y_{min} minimizing the R factor, that is minimizing

$$\frac{2^{n_i} - y}{2^{n_o} - y}\frac{1}{\sqrt{\frac{b^2}{n^{lsb2}} + \frac{a^2}{n^{lsb}}y + c^2 y^2}}. \tag{13}$$

The resulting equation is

$$y^3 + \left(\frac{a^2}{2c^2 n^{lsb}} - 2^{n_i}\right)y^2 + \left(\frac{2^{n_o}}{2n^{lsb}}\frac{a^2}{c^2} + \frac{2^{n_i+n_o}}{2n^{lsb}} - 3\frac{2^{n_i+1}}{n^{lsb}}\frac{a^2}{c^2}\right)y + \frac{2^{n_i+n_o}}{2n^{lsb}}\frac{a^2}{c^2} - \frac{2^{n_i} - 2^{n_o}}{n^{lsb2}}\frac{b^2}{c^2} = 0. \tag{14}$$

A graphical study of this equation shows there is one solution $0 \leq y_{min} \leq 2^{n_o}$, that is found analitycally through the Cardano method.

n_i	$\frac{n^{b,low}_{num}}{n^{lsb}}$	$\frac{n^{b,low}_{exa}}{n^{lsb}}$	$\frac{n^{b,low}_{app}}{n^{lsb}}$	$\frac{n^{b,up}_{num}}{n^{lsb}}$	$\frac{n^{b,up}_{exa}}{n^{lsb}}$	$\frac{n^{b,up}_{app}}{n^{lsb}}$
15.	60.4	61.1	47.0	4005.9	3878.1	4006.8
16.	161.2	161.6	154.2	3787.0	3899.1	3899.6
17.	393.0	393.3	389.4	3610.1	3664.1	3664.4
18.	1015.2	1014.2	1012.0	3017.0	3041.4	3041.8

n_i	α^{low}_{num}	α^{low}_{exa}	α^{low}_{app}	α^{up}_{num}	α^{up}_{exa}	α^{up}_{app}
15.	8.1	8.1	8.1	312.1	319.0	322.4
16.	16.6	16.6	16.6	304.9	313.1	313.9
17.	35.3	35.3	35.2	290.8	295.0	295.2
18.	84.8	84.7	84.7	243.8	245.8	245.8

Table 1
Compression parameter calculated for $n_o = 12$ bits, $R = 0.1$ and various n_i bits, with the numeric, exact and approximate analytic approaches.

n_i	$\frac{n^b_{min}}{n^{lsb}}$	α_{min}	R_{min}
15.	2098.5	15.35	0.001
16.	2058.6	31.1	0.004
17.	2039.7	62.7	0.015
18.	2030.4	125.9	0.059
19.	2025.8	252.3	0.221
20.	2023.6	505.0	0.723

Table 2
Compression parameter for $n_o = 12$ bits, minimizing R for various n_i bits calculated with the exact analytic approaches.

The compression parameters and the corresponding R_{min} are in Tab.2. Applying an approximate analytical approach similar to that leading to Eq.10, the result is $\frac{n^b_{min}}{n^{lsb}} \approx 2^{n_o-1}$, that is in good agreement with the exact analytical result.

1.5 Detector and compressor resolution

On the basis of the previous calculations we can consider the resolution of the detector, the quantization error, the quadratic sum of the two and the factor

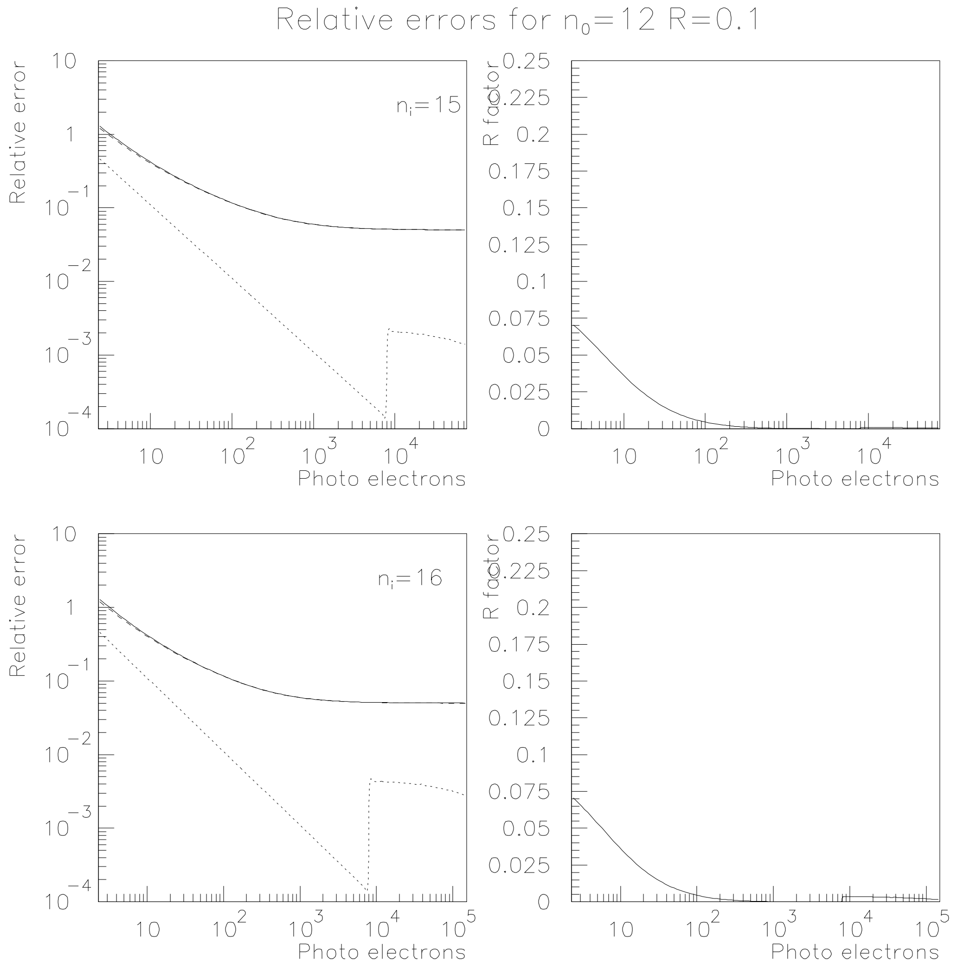

Fig. 2. Left: Relative resolution due to the detector (dashed), to the quantization (dash-dotted) and total(solid); Right: R factor versus photoelectrons for R_{min}, $n_i = 17$ and $n_i = 18$.

R versus the number of photoelectrons.

In Fig.2-3, we plot for different values of n_i and the corresponding R_{min} the relative contributions to the error due to the detector and to the quantization, as well the total and the R factor as defined in Eq.3.

2 Conclusions

With the previous calculations we have shown that a simple two linear compression function matches the expected resolution of the AUGER fluorescence detector with small degradation to quantization effects.

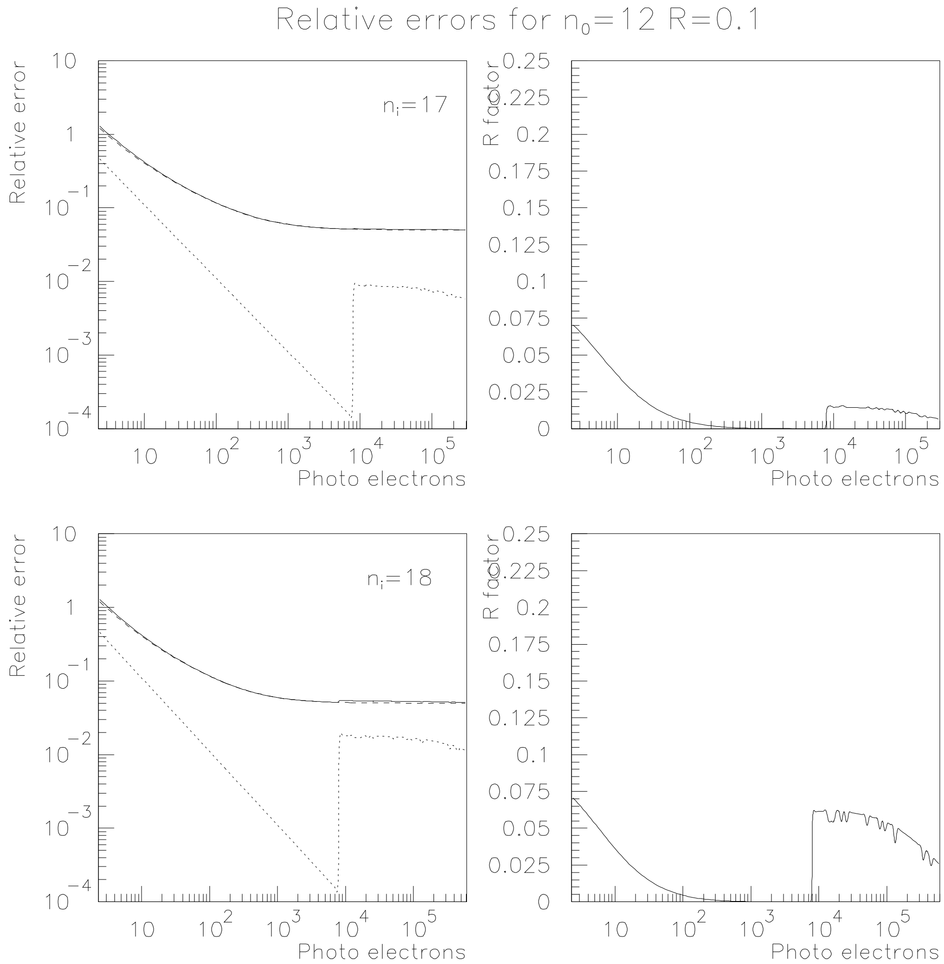

Fig. 3. Left: Relative resolution due to the detector (dashed), to the quantization (dash-dotted) and total(solid); Right: R factor versus photoelectrons for R_{min}, $n_i = 17$ and $n_i = 18$.

Using a 12 bit ADC, a range up to $2^{18} n^{lsb}$ can be measured without exceeding a 10% degradation of the precision.
If we are interested in a more restricted range of 16 bits, we can use up to 2 bits to improve the measurement of the noise. Therefore we can have $n^{lsb} = \frac{1}{4k} b$, that, for $R = 0.1$ and $b = 2.7$, gives $n^{lsb} = 1.08$. This extra resolution can be used to improve our capability of measuring the noise power spectrum still retaining the dynamic range deemed sufficient in [1].

References

[1] B. Dawson. Amplitude dynamic range in auger fluorescence electronics. GAP-97-064.

[2] A. Dell'Acqua et al. A digital front-end and readout microsystem for calorimetry at lhc. *IEEE Trans. Nucl. Scie.*, 40(4):516–531, 1993.

[3] B. Lofstedt et al. A digital front-end and readout microsystem for calorimetry at lhc. CERN-DRDC/90-74.

[4] V. G. Goggi. Dynamic range compression for calorimetry at lhc. Fermi Note # 2.