The anti-aliasing requirements for area and timing mesurements with the Fluorescence Detector read-out channel

P. W. Cattaneo

INFN Pavia, Via Bassi 6 - 27100, Pavia, Italy
Email: Paolo.Cattaneo@pv.infn.it

Abstract

This note calculates the requirements of the anti-aliasing filter for the read-out of the fluorescence detector of the AUGER experiment. The filter order and its cut-off frequency are related to the errors in area and timing reconstruction.

1 System transfert functions

The Fluorescence Detector read-out channel consist of a current preamplifier followed by a low pass anti-aliasing filter and a sampling ADC with period $T = 100ns$. The filter output signal should ideally satisfy the Nyquist criterion, that is band limitedness within $\omega_N = \frac{\pi}{2T}$ to allow shape reconstruction [1].
The input signal of the anti-aliasing filter $i(t)$ is supposed to be a rectangle with duration $\Delta > T$, (approximating the expected trapezoidal shape):

$$i(t) = \frac{Q}{\Delta}[1(t) - 1(t-\Delta)] \longleftrightarrow I(s) = \frac{Q}{s\Delta}[1 - \exp(-s\Delta)] \tag{1}$$

Three different n-th order anti-aliasing filters with the same cut-off frequency ω_c are analyzed: multiple RC, Bessel and Butterworth.
Their transfer functions are, respectively, [3]:

$$\begin{aligned}
H^n_{RC}(s) = &\ \tfrac{1}{D^n_{RC}(s)} = \frac{1}{(1+\frac{s}{\omega_{RC}})^n} & & \\
H^n_{Be}(s) = &\ \tfrac{1}{D^n_{Be}(s)} = \frac{1}{\sum_{k=0}^{n} b_k(\frac{s}{\omega_D})^k} & & b_k = \frac{n!/(n-k)!}{2n!/(2n-k)!}\frac{2^k}{k!} \\
H^n_{Bu}(s) = &\ \tfrac{1}{D^n_{Bu}(s)} = \frac{1}{\prod_{k=1}^{n}(\frac{s}{\omega_c} - s_k)} & & s_k = \exp(i\frac{2k+n-1}{2n}\pi)
\end{aligned} \tag{2}$$

Each of the previous filter type is described by its generic characteristic frequency ω_{Fi} and transfer function $H^n_{Fi} = 1/D^n_{Fi}$.
The relations between characteristic frequency and frequency cut is: $\omega_c = \omega_{Bu}$ for Butterworth, $\omega_c = \omega_{RC}\sqrt{2^{\frac{1}{n}} - 1}$ for multiple RC, while for Bessel, the relation emerges equating the module of the denominator to $\sqrt{2}$ and solving numerically the polynomial equation. Calling the solution $B_c(n)$, the relation is $\omega_c = B_c(n)\omega_D$.
The Laplace transform of the filter output signal $f(t)$ is therefore:

$$F(s) = \frac{Q}{s\Delta}[1 - \exp(-s\Delta)]\frac{1}{D^n_{Fi}(s)} \tag{3}$$

Since the system is expected to be causal, we can pass from the Laplace transform $F(s)$ to the Fourier transform $F(\omega)$ of the signal f(t) by substituting the variable s with $i\omega$ in the previous expression.

2 Area measurement

The signal bandlimited up to the Nyquist frequency [2] that interpolates the samples $f(nT)$ is $f_i(t)$

$$f_i(t) = \sum_{k=-\infty}^{+\infty} f(kT)\frac{\sin\omega_N(t - kT)}{\omega_N(t - kT)}. \tag{4}$$

An important quantity is the area of the sampled pulse A, that is easy to relate to the area, that is the charge Q, of the current pulse $i(t)$: recalling that the Fourier transform of a function evaluated at $\omega = 0$ equals its area, from Eq.1, $Q = I(0) = F(0)/H(0)$. $H(\omega)$ is assumed known with good precision (at least within the frequency cut) and can be measured with the help of a pulser.
The area can be estimated from the samples with [2]

$$\begin{aligned} F_*(\omega) &= \sum_{k=-\infty}^{+\infty} F(\omega + 2k\omega_N) \\ F_i(\omega) &= F_*(\omega)p_{\omega_N}(\omega) \end{aligned} \tag{5}$$

where $p_{\omega_N)}(\omega)$ is the rectangle function between $(-\omega_N, +\omega_N)$, $F_i(\omega)$ the Fourier transform of $f_i(t)$ and $F_*(\omega)$ the Fourier transform of the pulse train $Tf(kT)$. It is evident that $F_i(0) = F_*(0) = \sum_{k=-\infty}^{+\infty} Tf(kT)$.
In absence of noise, the only source of variability in the sum of the samples is the pulse phase (relative to the clock). It is important to note that the interpolation of the traslate is equal to the traslate of the interpolate, if and only if f is strictly satisfying the Nyquist criterion.

Defining $A^\tau = \sum_{k=-\infty}^{+\infty} Tf(kT+\tau)$ as the area estimator of the translate with phase τ:

$$A^\tau = F_*^\tau(0) = \sum_{k=-\infty}^{+\infty} f(kT+\tau) = \sum_{k=-\infty}^{+\infty} F(2k\omega_N)\exp(-i\tau 2k\omega_N)$$

and considering the average value of A^τ

$$\overline{A}^\tau = \frac{1}{T}\int_0^T d\tau F_*^\tau(0) = \frac{1}{T}\int_0^T d\tau \sum_{k=-\infty}^{+\infty} F(2k\omega_N)\exp(-i\tau 2k\omega_N) = F(0) = A$$

A^τ is an unbiased estimator of A.
The spread of A^τ is estimated as

$$\begin{aligned}
\sigma^2(A^\tau) &= \frac{1}{T}\int_0^T d\tau |F_*^\tau(0) - F(0)|^2 = \frac{1}{T}\int_0^T d\tau |\sum_{k\neq 0} F(2k\omega_N)\exp(-i\tau 2k\omega_N)|^2 \\
&= \frac{1}{T}\int_0^T d\tau \sum_{k\neq 0}\sum_{m\neq 0} F^*(2m\omega_N)F(2k\omega_N)\exp(-i\tau 2(k-m)\omega_N) \\
&= \sum_{k\neq 0} |F(2k\omega_N)|^2 \qquad \left(\frac{1}{T}\int_0^T d\tau \exp(-i\tau 2(k-m)\omega_N) = \delta_{km}\right).
\end{aligned}$$

There are three ways to estimate this series: considering the high frequency approximation, turning it into an integral or limiting it to its first term.
Within the first approach $|F(\omega)|^2 \leq 4(\frac{1}{b_{nFi}^n \omega_{Fi}})^2(\frac{\omega_{Fi}}{\omega})^{2(n+1)}$ for $\omega \gg \omega_{Fi}$, where b_{nFi} is the coefficient of the n-th term in $D_{Fi}(s)$; therefore

$$\begin{aligned}
\sigma^2(A^\tau) &\leq \frac{4}{b_{nFi}^{2n}\omega_{Fi}^2} 2\sum_{k\geq 0}\left(\frac{\omega_{Fi}}{2\omega_N k}\right)^{2(n+1)} = 2\frac{4}{b_{nFi}^{2n}\omega_{Fi}^2}\left(\frac{\omega_{Fi}}{2\omega_N}\right)^{2(n+1)} \sum_{k\geq 0}\frac{1}{k^{2(n+1)}} = \\
& 2\frac{4}{b_{nFi}^{2n}\omega_{Fi}^2}\left(\frac{\omega_{Fi}}{2\omega_N}\right)^{2(n+1)} \zeta(2(n+1)) \qquad (6)
\end{aligned}$$

where $\zeta(n)$ is the Riemann Zeta function and $\zeta(2n) = \frac{2^{2n-1}\pi^{2m}|B_{2m}|}{(2n)!}$ and B_n are the Bernoulli numbers [4]; for $n \geq 1$, $1.0 \leq \zeta(2(n+1)) \leq 1.1$, that is $\zeta(2(n+1)) \approx 1.0$, that means the series can be truncated at the first term, with little error. The weakness of this approach is that the validity range of the approximation depends on the filter type and may give a distorted view of the quality of the filter type and a gross error overestimation.
With the second approach the series can be approximates as

$$\sigma^2(A^\tau) = \frac{2}{2\omega_N}\int_{\omega_N}^{+\infty} d\omega |F(\omega)|^2. \qquad (7)$$

The delicate point is that the lower limit of this integral is rather arbitrary and that the result may depend strongly on it.

ν	0.30				0.60			
n	3	4	5	6	3	4	5	6
RC	1.00	0.51	0.32	0.21	5.80	4.50	3.90	3.44
Be	0.40	0.13	0.09	0.07	2.81	1.12	0.59	0.36
Bu	0.21	0.12	0.11	0.12	1.17	0.40	0.14	0.10

Table 1: $\epsilon(\%)$ from simulation for three types of filters, two values of $\nu = \omega_c/\omega_N$ and 4 filter orders.

The most reliable approach is considering only the first term of the series. For all transfert functions considered and all values $0 \leq \nu \leq 1.$, the series converges quickly. That is especially true for small ν and large n.
The relative error is therefore (for $\Delta \geq T$)

$$\epsilon = \frac{\sigma(A)}{\Delta} \leq \frac{\sqrt{2}}{\Delta} \frac{1}{(2\omega_N/2)} |H_{Fi}(2\omega_N)| \leq \frac{\sqrt{2}}{\pi} |H_{Fi}(2\omega_N)|. \tag{8}$$

The plots for RC, Bessel and Butterworth filters are, respectively, in Fig.1-2-3. It is important to realize that the errors reported in these plots are relative to $\Delta = T$ and that the actual errors scale with T/Δ, e.g. for $\Delta = 1\mu s$ ϵ is one tenth of what displayed in the plots.

2.1 Simulation of area measurement

The formulae developed for estimating the error on the area have been tested by running simulation of the readout electronics with different filters. In order to simulate the worst case a rectangular pulse has been used as current signal. Its width is $\Delta = 1.5T$, that because the presence of a sinus in the transform of the rectangle, gives "minima" suppressing area fluctuations for $\Delta = nT$ and maxima for $\Delta = (n + 1/2)T$. For comparison with the previous calculation the simulation results are multiplied by 1.5.
In Tab.1 the results of the simulation (corrected as explained) are shown. They can be compared to the calculations in Fig.2-3-**??**, remembering that the latter are to be considered as upper limits.

3 Timing measurement

Another critical parameter of the signal is the time baricenter (or timing). For a pulse $f(t)$ the timing is defined as

$$\tau_f = \frac{\int_{-\infty}^{+\infty} dt f(t) t}{\int_{-\infty}^{+\infty} dt f(t)}. \tag{9}$$

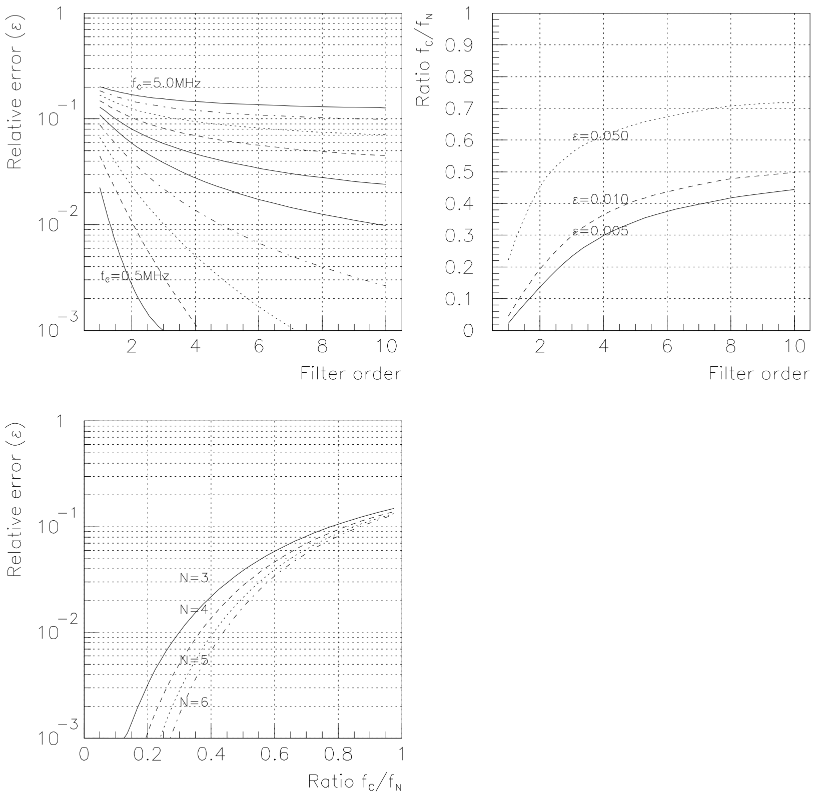

Figure 1: Multiple RC filter: (a) Area error versus filter order for a few normalized frequency cuts (b) Normalized frequency cut versus filter order for a few area errors (c) Area error versus normalized frequency cut for a few filter orders

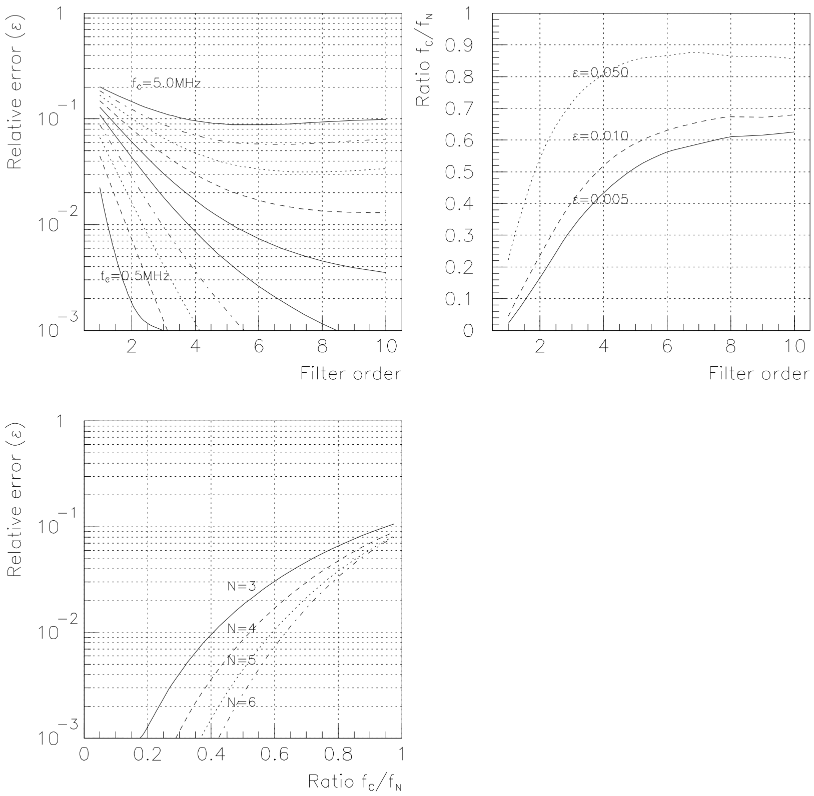

Figure 2: Bessel filter: (a) Area error versus filter order for a few normalized frequency cuts (b) Normalized frequency cut versus filter order for a few area errors (c) Area error versus normalized frequency cut for a few filter orders

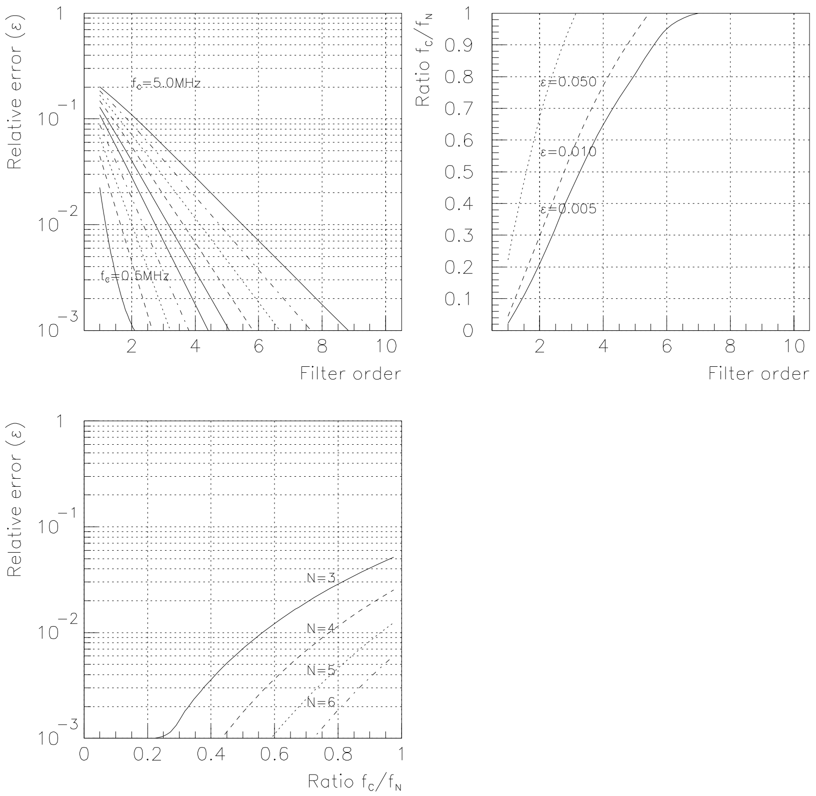

Figure 3: Butterworth filter: (a) Area error versus filter order for a few normalized frequency cuts (b) Normalized frequency cut versus filter order for a few area errors (c) Area error versus normalized frequency cut for a few filter orders

The relation between the timing of $f(t)$ and its Fourier transform is given by the well known formula

$$m_l = \int_{-\infty}^{+\infty} dt f(t) t^l = i^l F^{(l)}(0), \tag{10}$$

that allows to express the timing as $\tau_f = m_1/m_0 = iF^{(1)}(0)/F(0)$.
An estimator of τ_f from the samples is the normalized first moment of the samples η. Remembering that the Fourier transform of the samples is $F_*(\omega)$, then

$$F'_*(\omega) = \sum_k F'(\omega + 2k\omega_N) = -iT^2 \sum_k f(kT) k \exp(-ikT\omega), \tag{11}$$

and, considering the translate $f(t-\tau) \longleftrightarrow F^\tau(\omega)$

$$\begin{aligned} F_*^{\tau\prime}(\omega) &= (\sum_k F(\omega + 2k\omega_N) \exp(-i(\omega + 2k\omega_N)\tau))' \\ &= \sum_k F'(\omega + 2k\omega_N) \exp(-i(\omega + 2k\omega_N)\tau) + \\ &\quad \sum_k F(\omega + 2k\omega_N) \exp(-i(\omega + 2k\omega_N)\tau)(-i\tau), \end{aligned} \tag{12}$$

the expression for η is

$$\eta(\tau) = \frac{m_1}{m_0} = \frac{F_*^{\tau\prime}(0)}{F_*^\tau(0)} = i\frac{\sum_k F'(2k\omega_N) \exp(-i2k\omega_N\tau)}{\sum_k F(2k\omega_N) \exp(-i2k\omega_N\tau)} + \tau. \tag{13}$$

The average value $\overline{\eta} = (\eta(\tau) - \tau)$ can be obtained averaging the previous expression and assuming that the error on m_0 is small enough that the approximation $1/(1+x) \approx 1 - x$ applies

$$\begin{aligned} \overline{\eta} &= \frac{1}{T}\int_0^T d\tau(\eta(\tau) - \tau) = i\frac{F'(0)}{F(0)}\frac{1}{T}\int_0^T d\tau \frac{1 + \sum_{k\neq 0} \frac{F'(2k\omega_N)}{F'(0)} \exp(-i2k\omega_N\tau)}{1 + \sum_{k\neq 0} \frac{F(2k\omega_N)}{F(0)} \exp(-i2k\omega_N\tau)} \\ &\approx i\frac{F'(0)}{F(0)}\frac{1}{T}\int_0^T d\tau \left(1 + \sum_{k\neq 0}\left(\frac{F'(2k\omega_N)}{F'(0)} - \frac{F(2k\omega_N)}{F(0)}\right)\exp(-i2k\omega_N\tau)\right) \\ &= i\frac{F'(0)}{F(0)} = \tau_f. \end{aligned} \tag{14}$$

That proofs that, at least in the above approximation, $\eta(\tau)$ is an unbiased estimator of τ.
The error on timing can be written, under the same approximation, as

$$\sigma^2(\eta) = \frac{1}{T}\int_0^T d\tau(\eta - \overline{\eta} - \tau)^2$$

$$
\begin{aligned}
&= \frac{1}{T}\int_0^T d\tau \left|\frac{\sum_k F'(2k\omega_N)\exp(-i2k\omega_N\tau)}{\sum_k F(2k\omega_N)\exp(-i2k\omega_N\tau)} - \frac{F'(0)}{F(0)}\right|^2 \\
&= \int_0^T d\tau \left|\frac{F'(0)}{F(0)}\right|^2 \left|\frac{1+\sum_{k\neq 0}\frac{F'(2k\omega_N)}{F'(0)}\exp(-i2k\omega_N\tau)}{1+\sum_{k\neq 0}\frac{F(2k\omega_N)}{F(0)}\exp(-i2k\omega_N\tau)} - 1\right|^2 \\
&\approx \left|\frac{F'(0)}{F(0)}\right|^2 \frac{1}{T}\int_0^T d\tau \left|\sum_{k\neq 0}\left(\frac{F'(2k\omega_N)}{F'(0)} - \frac{F(2k\omega_N)}{F(0)}\right)\exp(-i2k\omega_N\tau)\right|^2 \\
&= \left|\frac{F'(0)}{F(0)}\right|^2 \sum_{k\neq 0}\left|\frac{F'(2k\omega_N)}{F'(0)} - \frac{F(2k\omega_N)}{F(0)}\right|^2 = \sum_{k\neq 0}\left|\left[\frac{F(\omega+2k\omega_N)}{F(\omega)}\right]'_{\omega=0}\right|^2 .
\end{aligned}
$$

Again the problem of estimating this series is best treated considering only the first term. Using the system transfert function in Eq.3, the relative timing error is

$$
\begin{aligned}
\epsilon^\tau &= \frac{\sigma(\eta)}{T} = \frac{\sqrt{2}}{|D_{Fi}(2\omega_N)|}\left|\frac{1}{2\pi}\left(\frac{\cos(\omega_N\Delta)(\omega_N\Delta) - \sin(\omega_N\Delta)}{\omega_N\Delta}\right) + \right. \\
&\left. \frac{\sin(\omega_N\Delta)}{\omega_N\Delta}\frac{\omega_N}{\pi}\left[-\frac{|D_{Fi}(2\omega_N)|'}{|D_{Fi}(2\omega_N)|} + i(\tau^H(0) - \tau^H(2\omega_N))\right]\right|, \qquad (15)
\end{aligned}
$$

where $\tau(\omega)$ is the filter delay for the frequency ω, in particular $\tau^H(0)$ equals b_1/ω_{Fi}, where b_1 is the coefficient of the first power term in $D_{Fi}(s)$. The dependance on Δ of this expression is much more complicated that in Eq.8, therefore it is useful to calculate approximate expressions more easily applicable in specific cases. For $\Delta >> T$, an upper limit on the error is

$$
\epsilon^\tau \leq \frac{\sqrt{2}}{|D_{Fi}(2\omega_N)|}\left[\frac{1}{2\pi}\right].
$$

An upper limit valid for every Δ can be obtained noting that both expressions containing Δ are in module smaller than or equal to 1. Hence it is

$$
\begin{aligned}
\epsilon^\tau &\leq \frac{\sqrt{2}}{|D_{Fi}(2\omega_N)|}\left[\frac{1}{2\pi} + Min(1, \frac{1}{\omega_c\Delta})\frac{\omega_c}{\pi}\right. \\
&\left.\sqrt{\left(\frac{1}{2\omega_N} + \frac{|D_{Fi}(2\omega_N)|'}{|D_{Fi}(2\omega_N)|}\right)^2 + (\tau^H(0) - \tau^H(2\omega_N))^2}\right], \qquad (16)
\end{aligned}
$$

alternatively a limit valid for $\Delta > T$ can be obtained substituting the expression $Min(\cdots)$ in Eq.16 with $1/\pi$.
Finally, it is of relevance the average error for an uniform distribution of Δ between T and some value $T_{max} \gg T$. Referring to Eq.15, the average on

ν	0.30				0.60			
n	3	4	5	6	3	4	5	6
RC	1.31	0.82	0.74	0.96	3.12	2.77	2.45	2.10
Be	0.80	0.54	0.83	1.05	2.05	1.00	0.40	0.42
Bu	0.29	0.48	0.91	1.28	0.98	0.59	0.48	0.67

Table 2: $\sigma(\tau)$(ns) from simulation for three types of filters, two values of $\nu = \omega_c/\omega_N$ and 4 filter orders.

the terms including $\sin(\omega_N\Delta)/(\omega_N\Delta)$ vanishes as $1/T_{max}$, while the average on the module of the $\cos(\omega_N\Delta)$ term is $2/\pi$.
The plots for these upper limits on the average timing error for RC, Bessel and Butterworth filters, respectively, are in Fig.4-5-6.

3.1 Simulation of timing measurement

The same simulations with $\Delta = 3T/2$ used for the measurement of the area are used to measure the timing of the signal. The error is the difference between the time centroid of the signal at the filter output and the normalized first moment of the samples.
The errors are reported in Tab.2 expressed in ns (or equivalently % of the period) and the prediction of Eq.15 are displayed in Fig.7-8-9

4 Conclusions

The performances of three anti-aliasing filter types are studied, relating the area and timing errors of the sampled waveform due to phase variation to the order and frequency cut of the filters. From this information the area and timing of the original current pulse can be easily recovered.
The main use of these calculations is helping designing filters that provide good antialiasing rejection still retaining as much as possible the original signal shape, that is with the highest possible bandwidth. The further choice of the filter type, namely Bessel and Butterworth, depends on the algorithm employed to reconstruct the original signal shape from the sampled waveform.
It is important to remind that the errors calculated are due to phase variation, and therefore amplitude independent, and do not include electronic noise and quantization. They will be dominant only for pulse amplitudes appropiately large.

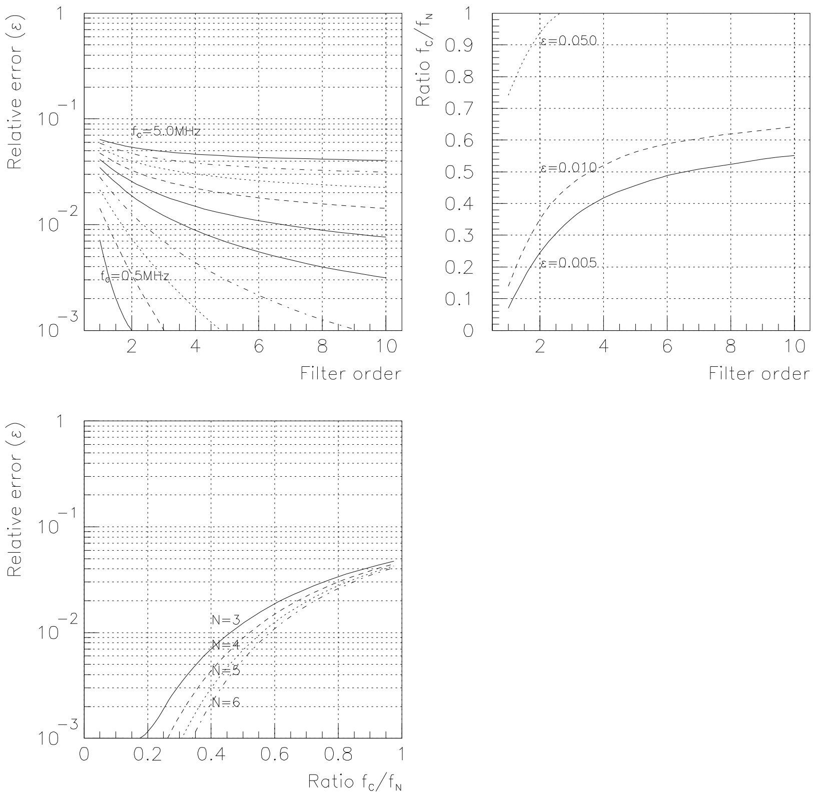

Figure 4: Multiple RC filter: (a) Timing error versus filter order for a few normalized frequency cuts (b) Normalized frequency cut versus filter order for a few errors (c) Timing error versus normalized frequency cut for a few filter orders

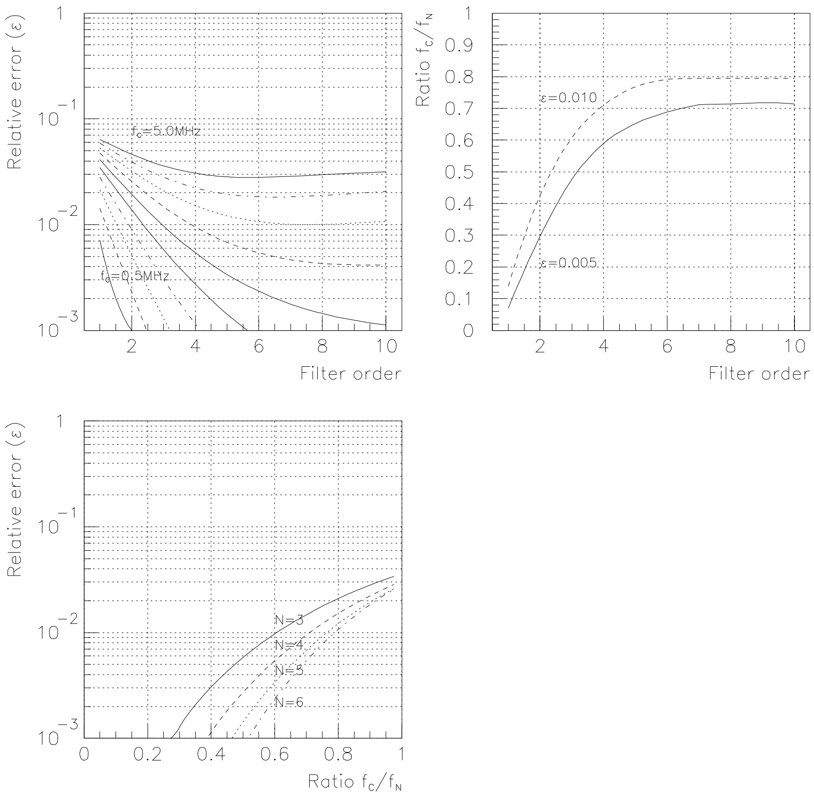

Figure 5: Bessel filter: (a) Timing error versus filter order for a few normalized frequency cuts (b) Normalized frequency cut versus filter order for a few errors (c) Timing error versus normalized frequency cut for a few filter orders

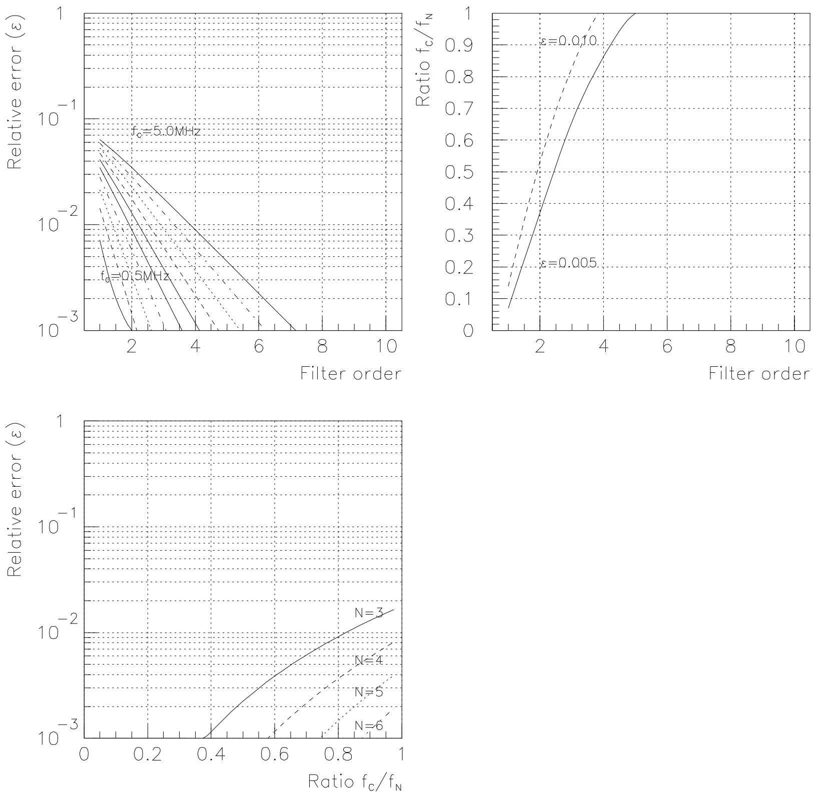

Figure 6: Butterworth filter: (a) Timing error versus filter order for a few normalized frequency cuts (b) Normalized frequency cut versus filter order for a few errors (c) Timing error versus normalized frequency cut for a few filter orders

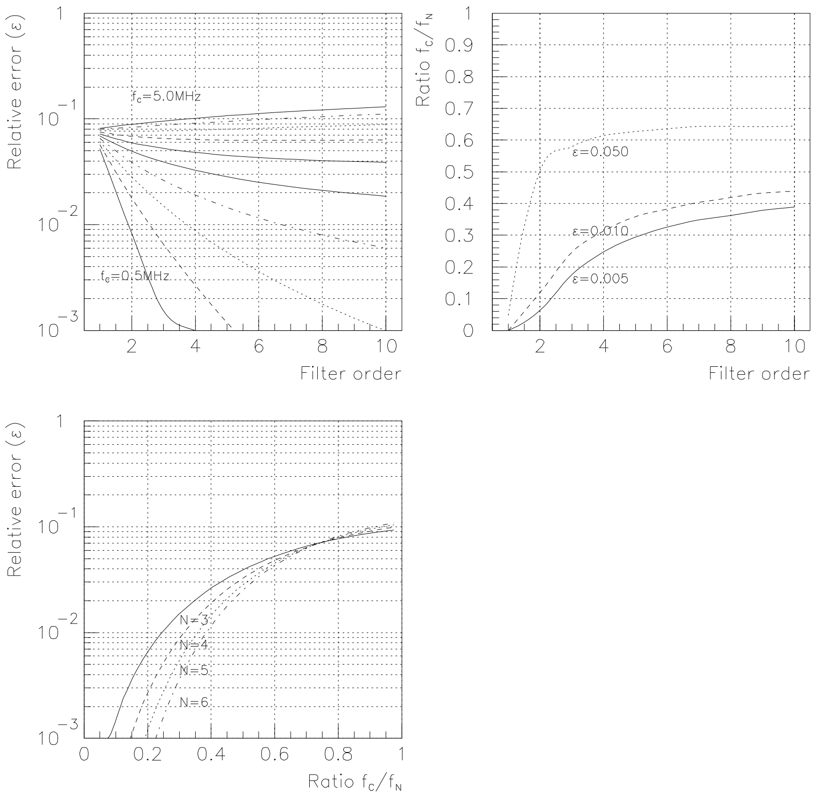

Figure 7: Multiple RC for $\Delta = 3T/2$: (a) Timing error versus filter order for a few normalized frequency cuts (b) Normalized frequency cut versus filter order for a few errors (c) Timing error versus normalized frequency cut for a few filter orders

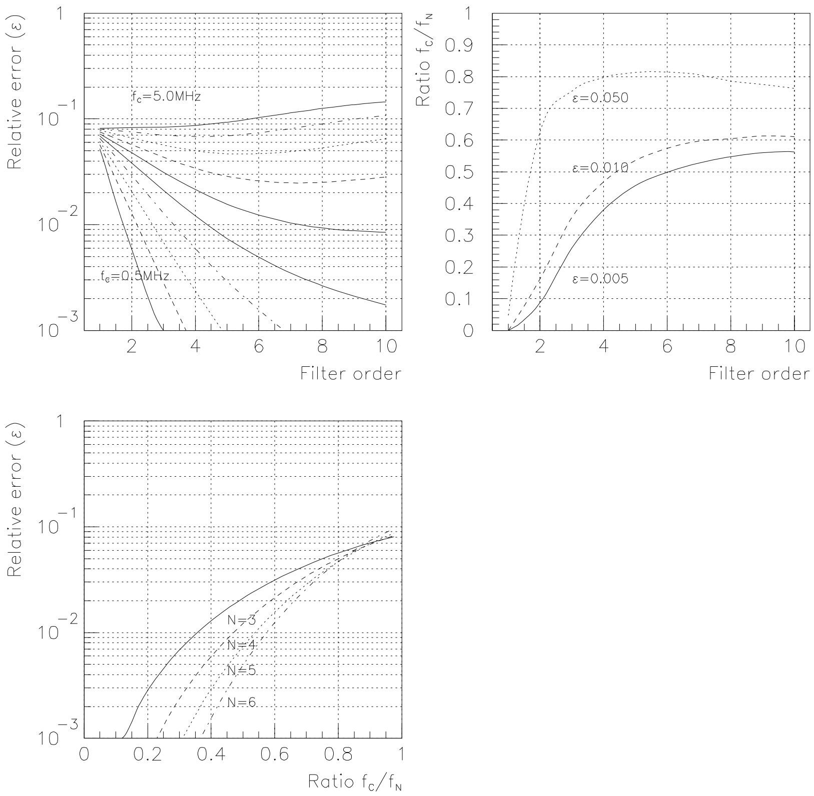

Figure 8: Bessel for $\Delta = 3T/2$: (a) Timing error versus filter order for a few normalized frequency cuts (b) Normalized frequency cut versus filter order for a few errors (c) Timing error versus normalized frequency cut for a few filter orders

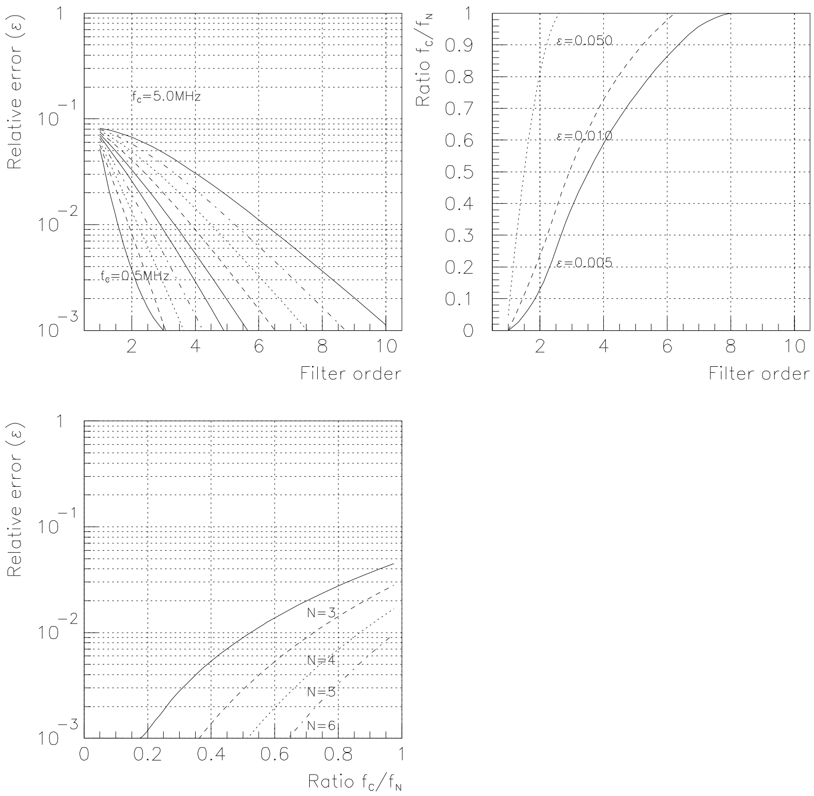

Figure 9: Butterworth for $\Delta = 3T/2$: (a) Timing error versus filter order for a few normalized frequency cuts (b) Normalized frequency cut versus filter order for a few errors (c) Timing error versus normalized frequency cut for a few filter orders

References

[1] S. Argiró et al. The analog signal processing system for the auger fluorescence detector prototype. To be published in IEEE Transaction on Nuclear Science.

[2] A. Papoulis. *Signal Analysis*, chapter 5. McGraw-Hill, New York, 1977.

[3] K.R. Laker R. Schaumann, M.S. Ghausi. *Design of analog filters*, chapter 1. Prentice-Hall Inc., Englewood Cliffs, NJ, 1990.

[4] F.Oberhettinger W. Magnus and R.P. Soni. *Formulas and Theorems for the Special Functions of Mathematical Physics*. Springer Verlag, Berlin, 1966.