The anti-aliasing requirements for amplitude mesurements with the Fluorescence Detector read-out channel

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Abstract

This note calculates the requirements on the anti-aliasing filter for the read-out of the fluorescence detector of the AUGER experiment. The filter order and its cut-off frequency are related to the error in amplitude reconstruction.

1 System transfert functions

The Fluorescence Detector read-out channel consist of a current preamplifier followed by a low pass anti-aliasing filter and a sampling ADC with period $T = 100ns$. The filter output signal should ideally satisfy the Nyquist criteria, that is band limitedness within $f_N = \frac{1}{2T} = \frac{\omega_N}{2\pi}$ to allow shape reconstruction [1].
The input signal of the anti-aliasing filter $i(t)$ is assumed to be a rectangle with duration $\Delta > T$, (approximating the expected trapezoidal shape):

$$i(t) = I_0[1(t) - 1(t-\Delta)] \longleftrightarrow I(s) = \frac{I_0}{s}[1 - \exp(-s\Delta)]. \qquad (1)$$

Three different n-th order anti-aliasing filters normalized at the same cut-off frequency ω_c are considered: a multiple RC, a Bessel and a Butterworth. Their transfer functions are, respectively, [4]:

$$\begin{aligned}
H^n_{RC}(s) &= \frac{1}{(1+\frac{s}{\omega_{RC}})^n} & & \\
H^n_{Be}(s) &= \frac{1}{\sum_{i=0}^n b_i(\frac{s}{\omega_D})^i} & b_i &= \frac{n!/(n-i)!}{2n!/(2n-i)!}\frac{2^i}{i!} \\
H^n_{Bu}(s) &= \frac{1}{\prod_{i=1}^n(\frac{s}{\omega_c}-s_i)} & s_i &= exp(j\frac{2i+n-1}{2n}\pi) \qquad (2)
\end{aligned}$$

The characteristic frequencies of the filters are $\omega_{RC} = \frac{1}{\tau_{RC}}$ and $\omega_D = \frac{1}{\tau_D}$, where τ_D is the delay of the filter.
The filter characteristic frequency coincides with its frequency cut only for the Butterworth filter. For the multiple RC the relation is $\omega_c = \omega_{RC}\sqrt{2^{\frac{1}{n}} - 1}$.
For the Bessel filter, the relation between τ_D and ω_c emerges equating the module of the denominator to $\sqrt{2}$ and solving numerically the polynomial equation to obtain the ratio of frequency cut and ω_D.
Defining $H(s)$ as the generic transfer function of the filter, the Laplace transform of the filter output signal $f(t)$ is therefore:

$$F(s) = \frac{I_0}{s}[1 - \exp(-s\Delta)]H(s) \tag{3}$$

Since the system is expected to be causal, we can pass from the Laplace transform $F(s)$ to the Fourier transform $F(\omega)$.

2 Amplitude measurement

The signal bandlimited up to the Nyquist frequency [3] that interpolates the samples $f(nT)$ is $f_i(t)$

$$f_i(t) = \sum_{n=-\infty}^{+\infty} f(nT)\frac{\sin\omega_N(t - nT)}{\omega_N(t - nT)}. \tag{4}$$

An estimator of the quality of the reconstruction of the shape before the sampling is the maximum error on the amplitude, when $f(t)$ is approximated by $f_i(t)$. The error $f(t) - f_i(t)$ is zero for $t = kT$ for construction; it can be shown [3] that, for any t:

$$|f(t) - f_i(t)| \le \frac{1}{\pi}\int_{|\omega|>\omega_N} |F(\omega)| d\omega = \frac{1}{\pi}\int_{|\omega|>\omega_N} I_0 \left|\frac{\sin(\frac{\omega}{2})}{\omega/2}\right| |H(\omega)|\, d\omega. \tag{5}$$

The expressions for $|H(\omega)|$ are simple for the n-th order multiple RC and Butterworth filters

$$\begin{aligned} |H^n_{RC}(\omega)| &= \frac{1}{(1 + (\frac{\omega}{\omega_{RC}}))^{\frac{n}{2}}} \\ |H^n_{Bu}(\omega)| &= \frac{1}{\sqrt{1 + (\frac{\omega}{\omega_c})^{2n}}} \end{aligned} \tag{6}$$

but complex and not very illuminating for the Bessel filter.
An error figure more relevant than Eq.5 is the ratio between the value in Eq.5 and the maximum of $f(t)$. The maximum is close to I_0 if the rise time is smaller than Δ, otherwise it is smaller. Because the filter is designed for anti-aliasing, its rise time cannot be shorter than T and under the assumption

$\Delta > T$, the filter rise time is shorter or comparable to T.
Within this approximation, the normalised error $\frac{|f(t)-f_i(t)|}{I_0}$ has a least upper bound ϵ given by

$$\begin{aligned}\epsilon = \sup\left(\frac{|f(t)-f_i(t)|}{I_0}\right) &\leq \frac{1}{\pi I_0}\int_{|\omega|>\omega_N} |F(\omega)| d\omega = \\ &= \frac{1}{\pi}\int_{|\omega|>\omega_N} 2\left|\frac{\sin(\frac{\omega\Delta}{2})}{\omega}\right| |H^n(\omega)|\, d\omega \leq \\ &\leq \frac{4}{\pi}\int_{\omega_N}^{+\infty} \frac{1}{\omega}\, |H^n(\omega)|\, d\omega \end{aligned} \tag{7}$$

From Eq.7, substituting $H^n(\omega)$ with the appropriate filter transfer function, the error, the filter order and the ratio between the cut-off frequency ω_c and the Nyquist frequency ω_N $\nu = \frac{\omega_c}{\omega_N}$ can be related for each filter type. An exact solution of Eq.7 is possible only for some $H^n(\omega)$ that are treated in the following.

2.1 Multiple RC

For the multiple RC filter the integral in Eq.7, setting $x = \omega/\omega_{RC}$, is

$$I_n(\nu) = \int_{1/\nu}^{+\infty} \frac{dx}{x(1+x^2)^{\frac{n}{2}}}. \tag{8}$$

Using results reported in [2], I_n can be calculated through

$$\begin{aligned} I_1(\nu) &= \int_{1/\nu}^{+\infty} \frac{dx}{x\sqrt{1+x^2}} = \frac{1}{2}\ln\left(\frac{\sqrt{1+\nu^2}+\nu}{\sqrt{1+\nu^2}-\nu}\right) \\ I_2(\nu) &= \int_{1/\nu}^{+\infty} \frac{dx}{x(1+x^2)} = \frac{1}{2}\ln\left(1+\nu^2\right) \\ I_k(\nu) - I_{k-2}(\nu) &= \frac{1}{k-2}\left(\frac{\nu^2}{1+\nu^2}\right)^{\frac{k-2}{2}}. \end{aligned} \tag{9}$$

More explicitly $I_n(\nu)$ can be written as follows, distinguishing n even or odd

$$\begin{aligned} I_{2k}(\nu) &= \frac{1}{2}\ln\left(1+\nu^2\right) - \sum_{i=2}^{k} \frac{1}{2i-2}\left(\frac{\nu^2}{1+\nu^2}\right)^{i-1} \\ I_{2k+1}(\nu) &= \frac{1}{2}\ln\left(\frac{\sqrt{1+\nu^2}+\nu}{\sqrt{1+\nu^2}-\nu}\right) - \sum_{i=1}^{k} \frac{1}{2i-1}\left(\frac{\nu^2}{1+\nu^2}\right)^{(2i-1)/2} \end{aligned} \tag{10}$$

and ϵ can be expressed as

$$\epsilon = \frac{4}{\pi} I_n(\nu). \tag{11}$$

An alternative approximate solution of Eq.7 can be obtained by approximating the derivative of the error with the difference in Eq.9

$$\begin{aligned} \Delta\epsilon\frac{\pi}{4} &= (I(n+1)-I(n-1))((n+1)-(n-1))/2 \\ \frac{d\epsilon}{dn} &\approx \frac{1}{2}\frac{4}{\pi}\frac{1}{n-1}\left(\frac{\nu^2}{1+\nu^2}\right)^{\frac{n-1}{2}}, \end{aligned} \tag{12}$$

the quality of the approximation improves with large n.
Treating n as a continuous variable, this equation can be integrated to give

$$\epsilon(n) = \frac{1}{2}\frac{4}{\pi}E_1\left(\frac{n-1}{2}\log\left(1+\frac{1}{\nu^2}\right)\right) \tag{13}$$

where $E_1(x)$ is a modified form of the exponential integral $Ei(x)$ [2], defined by

$$E_1(x) = -Ei(-x) = \int_x^{+\infty} dt\frac{\exp(-t)}{t} \qquad x > 0. \tag{14}$$

The main advantage of this approximate formulation is to evidentiate that the error depends on the combination $\frac{n-1}{2}\log(1+\frac{1}{\nu^2})$. Furthermore the problem of expressing $n(\nu,\epsilon)$ and $\nu(n,\epsilon)$ is reduced to the single problem of inverting $E_1(x)$.
The plots of the solution of Eq.11 are in Fig.1.

2.2 Bessel

For the Bessel filter the integral in Eq.7 cannot be calculated analitically, hence the integration is performed numerically.
The relevant plots are in Fig.2.

2.3 Butterworth

For the Butterworth filter, setting $x = \omega/\omega_c$, $y = x^n$ and using Eq.9 we have

$$\epsilon = \frac{4}{\pi}I_n(\nu) = \frac{4}{\pi}\int_{\frac{1}{\nu}}^{+\infty}\frac{dx}{x(1+x^{2n})^{\frac{1}{2}}} = \frac{4}{n\pi}\int_{\frac{1}{\nu^n}}^{+\infty}\frac{dy}{y(1+y^2)^{\frac{1}{2}}} = \frac{4}{2n\pi}\ln\left(\frac{\sqrt{1+\nu^{2n}}+\nu^n}{\sqrt{1+\nu^{2n}}-\nu^n}\right) \tag{15}$$

$$\nu = \left[\frac{(\exp(n\pi\epsilon/2)-1)^2}{4\exp(n\pi\epsilon/2)}\right]^{\frac{1}{2n}}. \tag{16}$$

The filter order can be expressed approximately, when $n\epsilon \ll 1$ and $\nu \le 1$, as

$$n \approx \frac{\ln\frac{\pi\epsilon}{4}}{\ln\frac{\pi\nu}{4}}. \tag{17}$$

The plots of the solution of Eq.15 are in Fig.3.

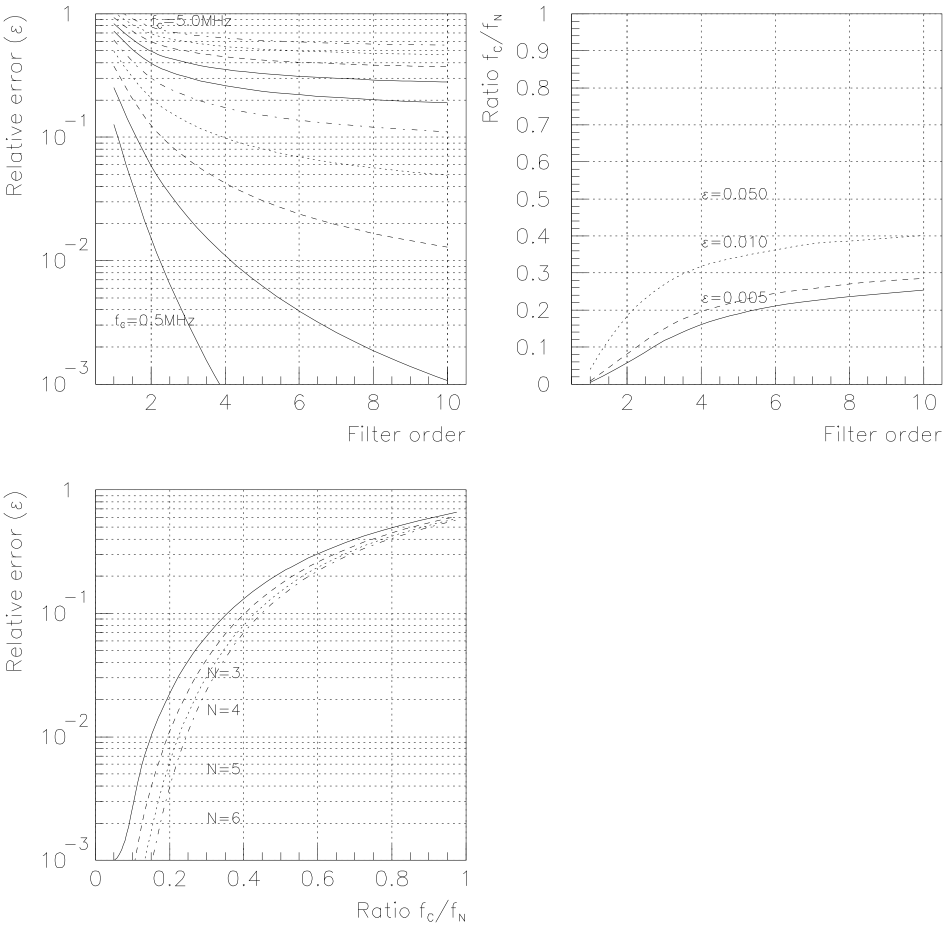

Figure 1: Multiple RC filter: (a) Error versus filter order for a few normalized frequency cuts (b) Error versus normalized cut for a few filter orders (c) Error versus normalized frequency cut for a few filter orders

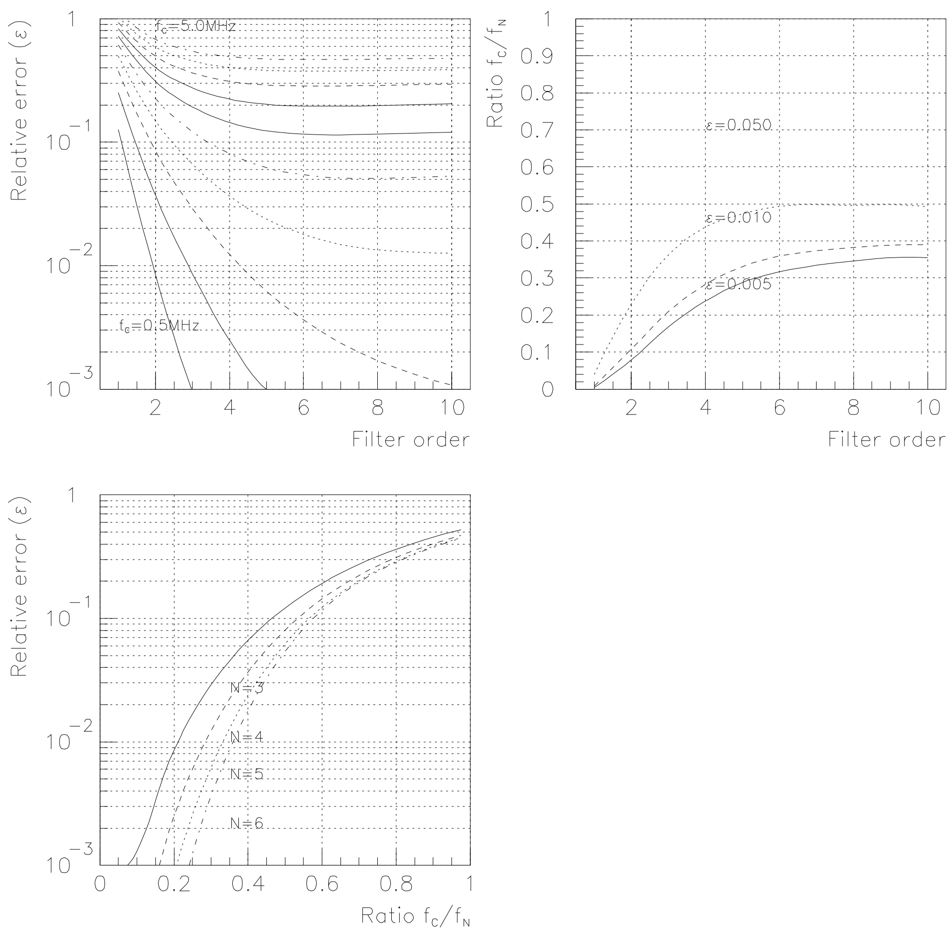

Figure 2: Bessel filter: (a) Error versus filter order for a few normalized frequency cuts (b) Error versus normalized cut for a few filter orders (c) Error versus normalized frequency cut for a few filter orders

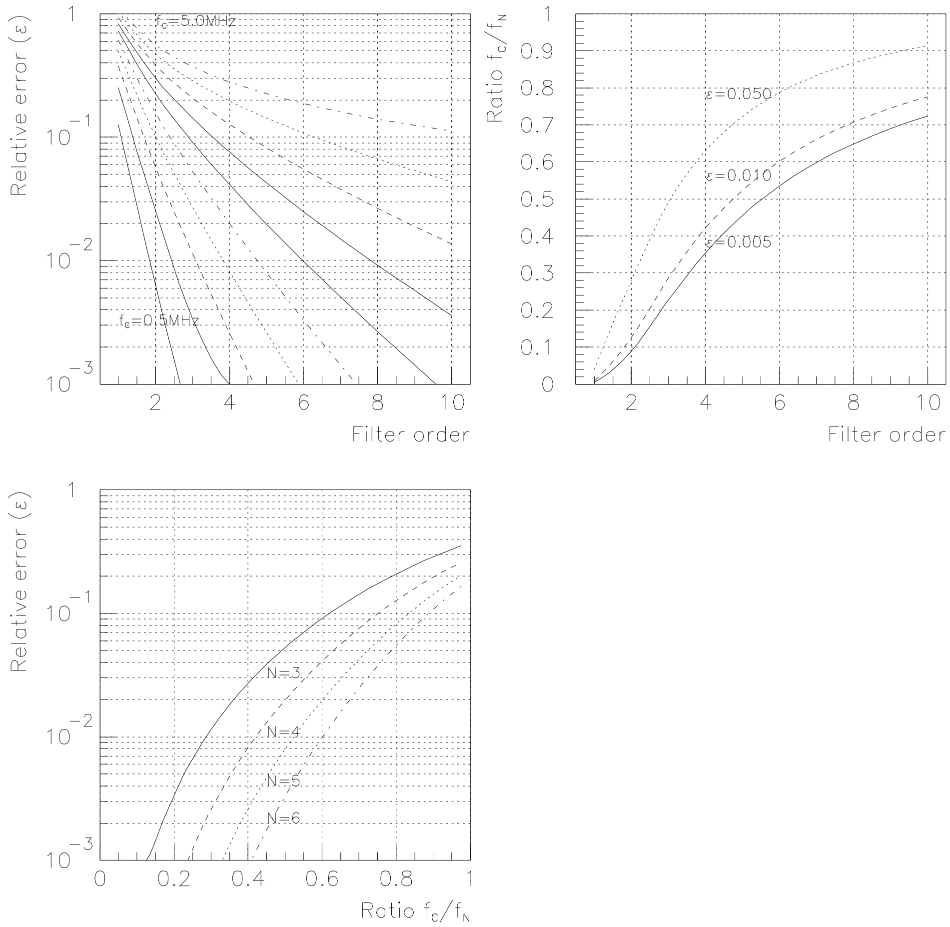

Figure 3: Butterworth filter: (a) Error versus filter order for a few normalized frequency cuts (b) Error versus normalized cut for a few filter orders (c) Error versus normalized frequency cut for a few filter orders

References

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